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# ALGEBRA

APPLIED TO

# GEOMETRY,

TO DETERMINE

THE POSITION OF A POINT AT REST, THE LOCUS OF A MOVING POINT, THE EQUATION TO THE STRAIGHT LINE, AND THE EQUATION TO THE CIRCLE.

BY X. Y. Z.

CAMBRIDGE: E. JOHNSON.
SOLD BY WHITTAKER & CO., AVE-MARIA LANE; AND
BELL & WOOD, FLEET STREET, LONDON.

1843.

CAMBRIDGE:
PRINTED BY METCALFE AND PALMER, TRINITY-STREET.

# PREFACE.

THE work now presented to the mathematical student was undertaken at the suggestions of some, who, having come up to the University unprepared, were frequently heard to complain of the want of *easy examples* by which to test their proficiency in this subject, or to rivet new principles and ideas in the memory. To endeavour to supply this want, as well as to explain fully and clearly the simpler principles and their application, has been the object aimed at.

The examples are framed so as to meet the case of the students alluded to, and of those whose knowledge does not extend beyond "Euclid's Elements," the *easier part* of "Trigonometry" and "Algebra," as far as *quadratic equations only*. If, however, the student have this knowledge, he may be assured that there is no problem proposed in the first and second chapters, for the solution of which he has not all the information that is requisite, so soon as he understands the terms made use of in the enunciations.

In Chapter I. he will find the problems useful, not only so far as they bear upon the present subject, but also from the fact, that almost every one depends upon results of Geometry, in the application of which he cannot have too much practice, if he would acquire dexterity in selecting out of them such as best suit any particular case.

Chapter II. contains a class of problems which are thought not to be favourites with the generality of students, partly because few or none are proposed to them at the commencement of the subject; they are afterwards neglected, and when by chance met with, seem to be fraught with difficulty; whereas a few hours devoted to them at a proper time would suffice to shew that such questions may be classed among the easiest.

Chapter III. treats of the Equation to the Straight Line, and contains a variety of examples on intersections, perpendiculars, &c., with figures to be traced from the equations to their sides. It was thought that the student, after having described a few of them, would inevitably get into the habit of recognizing, at a glance, the circumstances of position and inclination, so as to *see* a line in its equation, a figure in a set of equations, and insensibly pass on to the idea of a curve being represented by an equation.

Chapter IV. treats of the Circle. In obtaining the equation to the tangent, Descartes' method is preceded by that in which the known properties of the circle are referred to; this method being more convenient when the centre of the circle is not the origin. By a reference to these the equation to the secant is also more easily obtained, and by a process which cannot lead to confusion: the equation to the tangent is introduced and eliminated, and thus $(x', y')$, the point of contact, disappears from the equation quite naturally. But to pass from $hx'+ky'=r^2$ to $hx+ky=r^2$, by the considerations usually employed, seems to lead students into the error of confounding one line with the other.

To satisfy the student that these equations might be obtained without reference to Geometry, the properties referred to, and several of the more important propositions in Euclid's third book, are established by processes purely analytical, even without employing a figure. These demonstrations ought, perhaps, to have preceded the propositions on the tangent, &c. The novelty (to the student) of arriving at known geometrical truths by a new route, would perhaps more forcibly impress him with the justness of the principles which direct him, and beget a feeling of safety in their further application: but it was thought best to put him in possession of the results with the least delay, by means of previously acquired knowledge. How this knowledge may be arrived at differently becomes afterwards the matter of curiosity, which he will be able to gratify without retarding his progress.

*Cambridge*, 1843.

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## CHAPTER IV.

### *On the Circle.*

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## ERRATA.

The Student is requested to correct the following Errata.

Page 7, line 16, *for* ($a$) *read* ($\delta$).
" 8, " 5, same.
" 8, " 24, This equation should be followed by ...... ($\delta_2$).
" 8, " 34, " " " ...... ($\delta_3$).
" 9, " 27, *for* ($a$) *read* ($\delta$).
" 9, " 34, *for* assumes *read* assume.
" 13, " 6, *for* $2pxzx$ *read* $2przx$.
" 34, " 11, *for* greater *read* less.
" 30, " 25, *for* when in *read* where it cuts.
" 38, " 23, *for* highest *read* lower.
" 38, " 25, *for* lower *read* higher.
" 56, " 3, *for* $x = -a,\ y = b_2,$ *read* $y = b_2,\ x = -a$.
" 71, " 3, *for* $2x6$ *read* $2x-6$.

[Being pressed for room, it was found impossible to insert more than the Answers to the Loci. Students are informed that a Key is prepared, and will be put to press so soon as it is ascertained that they would think their progress facilitated by a publication of the solutions.]

# ALGEBRA APPLIED TO GEOMETRY.

## CHAPTER I.

### 1. *Position of a Point in a Plane.*

BEFORE introducing to the student, terms and their definitions, he is requested to consider what he would do, were he asked *to determine the position of a point in a plane.*

He would first assure himself that he understands the proposition, and to this end recal the definition of a plane, viz. "a plane is a superficies or surface such, that if any two points whatever be taken in it, the right line between them lies wholly in that superficies;" in more common words then, the proposition is *to determine the position of a point in a flat surface.*

A flat surface has length and breadth, and the proposition does not state that these are limited, or that the surface is placed in any particular position. Being, then, at liberty to make a supposition, he would perhaps, as the most convenient case, conceive that he is standing upon the surface, that it is extended indefinitely *in all directions* (*right, left, &c.*); and that a point being situated *any where* in that surface, it is required of him *to determine its position.*

Now he is aware that the position of a point might be called determined, if *he knew* where to look for it. This would lead him *to refer the position of the point to his own.* Farther, he would reflect, that in order to *find* any point, he would *require* to know *its distance from him,* and *the direction* in which that

distance lies, (viz. either to the right or left, &c.); hence, generalizing, he would conclude, that the way to determine the position of a point in a plane is *to refer it to another point, and to determine both its distance and direction with respect to that point.*

### 2. *Position of a point expressed Algebraically.*

Having, as above, settled in his own mind how the position of a point is determined, there would appear but little difficulty in the following proposition, *to represent algebraically the position of any point in a plane.*

Here it is obvious, that the position of a point is to be assumed as known, (*i. e.* its distance from and direction with respect to another point are to be supposed ascertained,) or at least ascertainable; and then all that the proposition requires is to express these in a general manner by means of the Symbols of Algebra.

The student, taking in the plane that point over which he stands as the point of reference, would first consider the (to him) familiar directions right, left, backwards and forwards; these directions lying in two straight lines which intersect each other at right angles. To proceed methodically, he would take a sheet of paper to represent the plane, and drawing two lines $X'OX$, $YOY'$, (fig. 1) intersecting at right angles in the point $O$, assume $O$ as the point on which he stands, and $OX$, $OX'$ to be respectively right and left, and $OY$, $OY'$ forwards and backwards.

It would not require much consideration for him to discover, that to represent any point on either of these lines, it would be sufficient to have two *different symbols* to express any distance, one *for each line.* Thus

$x$ might be taken to represent any distance along $XX'$
$y$ .......................................... $YY'$;

and applying the principle already explained to him in Trigonometry, viz. that *contrariety of position is represented by contrariety of sign,* he would easily represent the position *with respect to the point $O$* of any point on either of the lines $XX'$, $YY'$; for

assuming *right* and *forwards* to be the *positive* directions, and *left* and *backwards* the *negative*, then

$+x$ or $x$ might represent any point in $OX$,
$-x$ ...................... $OX'$,
$+y$ or $y$ ...................... $OY$,
$-y$ ...................... $OY'$.

Here it is to be observed, that $x$ and $y$ are taken in the general, indeterminate sense which the student has been accustomed to attach to those symbols in the algebraical problems which he has worked, viz. as representing in a general manner quantities which are indeterminate, and the value of which it is the object of a proposition or problem to determine: with this distinction however, that when applied to geometry as above, these quantities represent lines of indefinite magnitude; whereas, hitherto he has most probably only used them as the representatives of *any* indeterminate *numbers*.

The above principles, with a trifling variation to the meaning of the letters $x$, $y$, are sufficient to represent the position of a point in any other direction. For let it be assumed that $x$, $y$, also represent distances measured *parallel to* $X'X$ and $YY'$, *as well as along* those lines.

Let $P_1$ (fig. 2) be a point not on either of the lines $X'X$, $YY'$; draw $P_1N$ parallel to $OY$, and draw the dotted line $OP_1$.

Let the student consider how he might *direct* a person at $O$ to the point $P_1$.

In two ways, either by giving the distance $OP_1$ and the angle which that line makes with $X'X$, or, which will appear more simple, by *giving the two distances which lead to it.* Assuming then, since $x$ and $y$ may represent *any* distances, that $x$ represents $ON$ and $y$ $NP_1$, the point $P_1$ might be represented algebraically by $(x, y)$.

This method admits of applying the principle of signs, and it will soon be evident that, by using these, the position of any point, in either of the other compartments in which the lines $X'X$, $YY'$ divide the plane, may be properly represented.

Thus let $P_2$ (same figure) be a point at the same distance from $O$ and the line $X'X$ as $P_1$, then $P_2N = P_1N$, and will be represented by $(-y)$ to indicate the difference of position. Hence the

point $P_2$ will be represented by $(x, -y)$. And for similar reasons the points

$P_3$ (fig. 3) will be represented by $(-x, y)$
and $P_4$ ........................ $(-x, -y)$.

This method of representing the position of a point in a plane, at which the student has been supposed to arrive of himself, is that employed in the application of Algebra to Geometry. The lines, &c. to which the symbols refer have received certain names as under.

$O$, the point in which the lines $X'X$, $YY'$ intersect, and from which the measures or distances that determine the position of any point originate, is called *the origin.*

$x$, $y$, the distances *with their proper signs* are called *the co-ordinates* of the point $P$.

$X'X$, $YY'$, the lines along or parallel to which the co-ordinates are drawn are called *the co-ordinate axes :*

$X'X$ being distinguished as the axis of $x$
$YY'$............................ $y$,

it is usual to call the portions $OX$, $OY$, the positive axes
........................ $OX'$, $OY'$, the negative axes,

and then a coordinate is positive or negative according as it is drawn parallel to, or along a positive or negative axis.

Also for distinction, $y$ is called the ordinate, and
............ $x$ ...... the abscissa.

When, as in the present case, the co-ordinate axes intersect at a right angle, they and the co-ordinates also are called *rectangular;* and when inclined to each other at an angle > or < 90°, they are called oblique axes, and the co-ordinates are also said to be oblique. But as it is always understood that $x$ and $y$ or the co-ordinates are distances taken along or parallel to the co-ordinate axes, it is obvious that the position of a point may be represented by the same symbols when the axes are oblique as when they are rectangular.

The student has seen that when a point is situated not on either of the axes, one co-ordinate is sufficient to represent its position; but, that two co-ordinates are necessary when the point is situated anywhere between the axes. It will, however,

be better at all times to represent or give the position of a point by means of *two symbols*, with the understanding that *the first refers to the axis of $x$, the second to the axis of $y$*, and then when a point is on either axis, $o$ will stand in the place of the unnecessary co-ordinate.

This being agreed, it is known directly that if a line be said to pass through a given point $(o, d)$, it is meant that it cuts the axis of $y$ in a point distant $d$ from the origin; so $(-g, o)$ represents a given point in the negative axis of $x$ distant $g$ from the origin. And if it were required to represent geometrically a point, the position of which is expressed algebraically by $(b, 3b)$, the student would have to take along the axis of $x$ a distance $b$, and from the extremity of this a distance $3b$, in a direction parallel to $OY$, as in (fig. 4), where $ON=b$, and $NP=3b$, and $P$ is the point required.

The above observations the student will find it useful to bear in mind whenever he has to adapt an equation to a particular case by *the mere substitution of co-ordinates*. The method of two symbols has besides the advantage of establishing a relation between the co-ordinates and the axes when the former are represented by letters, which cannot without surcharging the memory with conventional rules be restricted to one axis rather than the other.

### 3. *Polar Co-ordinates.*

It was said in the last article that the point $P$ might be indicated by means of its distance from $O$ and the angle which that distance makes with the line $X'X$. Hence another method of representing the position of a point, viz. by its *polar co-ordinates*, sometimes employed in cases where the motion of a point *revolving about another* is investigated.

In this method the line $YY'$ is dispensed with. $X'X$ is supposed given in position, and a point $O$ in $X'X$ is taken as the point or centre of revolution. This point (fig. 5) is called *the pole*, the line $OX$ *the initial line*, the distance $OP_1$ *the radius vector* of the point $P_1$, and the angle $P_1OX$ the *angle of revolution.*

The *radius vector* and *angle of revolution*, which when their magnitudes are indeterminate are commonly represented by $\rho$ and $\theta$ respectively, are called *the polar co-ordinates* of any point in the plane in which $X'X$ lies.

The *initial position* of the radius vector is that which it is assumed to have *when* $\theta = 0$. If then $(\rho, \theta)$ be the polar co-ordinates of $P_1$, $P_1OX = \theta$, and the radius vector in its initial position coincides with $OX$. It may be supposed to have come into the position $OP_1$ by revolving from right to left in a direction opposite to that of the hands of a watch. These two suppositions are those most commonly made: it must however be observed, that both the initial position and direction of revolution are quite arbitrary, and when not specified in a question, may always be selected with a regard to convenience.

It may be as well to recall to the student's memory the convention observed in measuring angles, viz. that when one direction of revolution is assumed as positive, the *contrary* is considered *negative*. Thus, if $P_1P_2$ had both started from the same point $p$ in $OX$, and revolved at the same rate but in opposite directions about $O$, the polar co-ordinates of $P_1$ would be $(\rho, \theta)$, those of $P_2$ $(\rho, -\theta)$.

In the application of Algebra to Geometry rectangular axes and co-ordinates are usually employed, those in the generality of cases leading to simpler results. When therefore it is not otherwise specified, the student will consider that in the problems and propositions which follow, points, lines, &c. are referred to rectangular axes.

When the axes are oblique, the angle of inclination is either given or left to be assumed at pleasure.

When a proposition is to be investigated in terms of polar co-ordinates, the initial position of the radius vector is either given or left to be assumed at pleasure.

PROP. 1. To find the distance between two points in terms of their co-ordinates.

Let $OX$, $OY$, (fig. 6) be the co-ordinate axes to which the points $P$, $Q$ are referred, and let $x_1$, $y_1$, be the co-ordinates of $P$, $x_2$, $y_2$, the co-ordinates of $Q$.

Draw $PN$, $QM$ parallel to $OY$; $Qm$ parallel to $OX$; then $QmP$ is a right angle. Join $P$, $Q$, and let $PQ = d$ the required distance.

From the triangle $QmP$

$$(QP)^2 = (Pm)^2 + (Qm)^2.$$

Now the figure $QmNM$, is a parallelogram,

$$\therefore\ Qm = MN = ON - OM = x_1 - x_2,$$

and $$Pm = PN - mN = PN - QM = y_1 - y_2.$$

Hence, by substitution,

$$d^2 = (y_1 - y_2)^2 + (x_1 - x_2)^2,$$

$$\therefore\ d = \sqrt{\{(y_1 - y_2)^2 + (x_1 - x_2)^2\}} \ \ldots\ldots\ldots (\delta),$$

using the positive sign with the radical, since the absolute distance only is required.

From this expression, which the student must impress on his memory, *others may be obtained* to suit particular cases depending upon different positions of the second point, by *substituting* instead of $x_2$, $y_2$, *the co-ordinates which determine those positions.* Thus,

COR. 1. Suppose the second point were situated in the positive axis of $x$ at a distance $a$ from the origin, its co-ordinates would be $(a, o)$, and substituting these for $(x_2, y_2)$ in $(a)$ its distance from $P$ would be

$$d = \sqrt{\{(y_1 - o)^2 + (x_1 - a)^2\}},$$

$$\text{i.e. } d = \sqrt{\{y_1^2 + (x_1 - a)^2\}};$$

a point in the positive axis of $y$ at distance $b$ from the origin would have $(o, b)$ for co-ordinates, and its distance from $P$ would be

$$d = \sqrt{\{(y_1 - b)^2 + (x_1 - o)^2\}}$$
$$= \sqrt{\{(y_1 - b^2) + x_1^2\}};$$

and if we supposed a point to coincide with the origin, since its co-ordinates would be $(o, o)$, therefore the distance of $P$ from the origin is

$$d = \sqrt{\{(y_1 - o)^2 + (x_1 - o)^2\}}\ \cdot$$
$$= \sqrt{(y_1^2 + x_1^2)}.$$

COR. 2. Let the distances of $Q$ (fig. 7) from the axes of $y$ and $x$ be $a$, $b$ respectively; its co-ordinates will be $(-a, b)$, to indicate that it is on the left of the origin. Let its distance from $P$ $(x_1, y_1)$ be required.

Construct as in figure, and let $QP = d$; then from right-angled triangle $PmQ$

$$QP^2 = Qm^2 + Pm^2;$$

but $$Qm = MN = NO + MO = (x_1 + a),$$
$$Pm = PN - mN = PN - QM = (y_1 - b).$$

Hence, substituting in the first equation, and extracting

$$d = \sqrt{\{(y_1 - b)^2 + (x_1 + a)^2\}};$$

if now $(-a, b)$ be substituted for $(x_2, y_2)$ in $(a)$, the result is

$$d = \sqrt{[(y_1 - b)^2 + \{x_1 - (-a)\}^2]};$$

which will evidently become the same as the last if the sign of direction be subject to the ordinary algebraical rule. Hence the student must bear in mind, that *the signs of direction are subject to the ordinary rules of Algebra,* when the quantities which they affect are connected by the same signs used in the algebraical sense.

Prop. 2. Two points being referred to oblique axes, to find an expression for their distance from each other.

Let $OX$, $OY$, (fig. 8) be the co-ordinate axes; $i = YOX$ the angle of inclination; $P$, $Q$ the points whose co-ordinates are $(x_1, y_1)$, $(x_2, y_2)$ respectively; draw $PM$, $QM$ parallel to $OY$, $Qmn$ parallel to $OX$, then $Pmn = YOX = i$, and from the triangle $QmP$ where the angle $QmP = (\pi - i)$,

$$QP^2 = Pm^2 + Qm^2 - 2PmQm \cos(\pi - i);$$

but $$\cos(\pi - i) = -\cos i;$$

also $$Pm = (y_1 - y_2),\ Qm = (x_1 - x_2),$$

and let $QP = d$; then substituting

$$d^2 = (y_1 - y_2)^2 + (x_1 - x_2)^2 + 2(y_1 - y_2)(x_1 - x_2)\cos i.$$

The observations in Cor. 1 and 2 are applicable to this case.

Prop. 3. To express the distance between two points in terms of their polar co-ordinates.

Let $O$ (fig. 9) be the pole, $OX$ the initial line, $P_1P_2$ the points whose polar co-ordinates are $(\rho_1, \theta_1)$, $(\rho_2, \theta_2)$, respectively. Join $P_1P_2$, then the angle $P_1OP_2 = (\theta_1 - \theta_2)$, and from the triangle $P_1OP_2$

$$(P_1P_2)^2 = OP_1^2 + OP_2^2 - 2OP_1 \,.\, OP_2 \cos(\theta_1 - \theta_2),$$

or if $P_1P_2 = d$,

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\theta_1 - \theta_2).$$

So soon as the student has possessed himself of the principles just laid down, he is advised to apply them to the problems which follow, which have all been framed so as to involve no geometrical or other difficulties. The properties of the right and oblique-angled triangle, those of lines touching and cutting a circle, and of those intersecting within a circle, with the theory of proportionals, are all the results in the "elements" for which he need tax his memory. Trigonometry is little more than just alluded to; and all that is required of him as an algebraist, is to be able to manage simple and quadratic equations of the easiest kind.

In setting to work, the first step in most cases will be, to draw a figure, and to write against each line the algebraical symbol fixed upon to represent it.

Next, from the geometrical relations between the lines to form as many equations between the symbols as there are unknown quantities.

Having done this, the student has no farther use for the figure, with which therefore he need no longer trouble himself, but proceed to determine the values of the unknown quantities by processes purely algebraical, viz. elimination, substitution, extraction, &c.

## EXAMPLES.

The co-ordinates of the angular points of a triangle are $(a, b)$, $(c, d)$, and the length of the perpendicular from third angular point on opposite side is $p$; required the area of the triangle.

Here a figure is unnecessary.

The area of a triangle $= \frac{1}{2}$ (base $\times$ altitude), and taking one side as base, the altitude is the same as the perpendicular from opposite angular point. In the present case then, the base will equal the distance between the points $(a, b)$, $(c, d)$, which by $(a)$

$$= \sqrt{\{(a-c)^2 + (b-d)^2\}},$$

also altitude $= p$. Hence

area of the triangle $= \frac{1}{2}\,[p\,\sqrt{\{(a-c)^2 + (b-d)^2\}}]$.

Find the area of a quadrilateral figure, the co-ordinates of whose angular points are $(o, b)$, $(a, o)$, $(c, d)$, $(e, f)$, and describe it when $d < b$, $e > a$ and $< c$, $f > b$.

Take (fig. 10) $OD$ which assumes $= b$,

$OA$ ............ $= a$,

$ON$ ............ $= c$,

$NB$ ............ $= d$,

$Om$ ............ $= e$,

$mC$ ............ $= f$;

then the figure $ABCD$ is the parallelogram required. Through $B$ draw $NBP$ parallel to $OY$, and through $C$ draw $PCn$ parallel to $OX$; then

$$\begin{aligned} \text{area } ABCD &= ONPn - \triangle BAN - \triangle BCP - \triangle CDn - \triangle OAB \\ &= ON.PN - \tfrac{1}{2}(BN.AN + BP.CP + Cn.nD + DO.OA) \\ &= ef - \tfrac{1}{2}\{d(c-a) + (f-d)(c-e) + e(f-b) + ab\}. \end{aligned}$$

A circle (radius $= r$) is described touching the positive axis of $x$, and from the extremity of a vertical diameter a chord, whose length $= r$, is drawn and produced to cut the axis of $x$ at the origin; find the length of the line so produced, and the distance of the point of contact from origin.

Let (fig. 11) $OX$, $OY$ be the co-ordinate axes, $BDA$ the circle touching the axis of $x$ in $A$, $BD$ the chord $= r$, which being produced meets the axis of $x$ at $O$.

Let $\left.\begin{aligned} \lambda &= BO, \\ \delta &= OA, \end{aligned}\right\}$ which are required;

from Euclid (38, 3), $\quad \delta^2 = \lambda(\lambda - r),$

$$= \lambda^2 - \lambda r;$$

but from the right-angled triangle $BAO$,

$$\delta^2 = \lambda^2 - 4r^2, \quad \because BA = 2r:$$

equating these two values of $\delta^2$,

$$\lambda^2 - 4r^2 = \lambda^2 - \lambda r,$$

$$\therefore \lambda = 4r,$$

and $\quad \delta^2 = 16r^2 - 4r^2,$

$$\therefore \delta = \sqrt{(12)}\,.\,r.$$

A circle is described touching the axis of $x$, and from the extremity of the vertical diameter and from the centre two lines are drawn to the origin. The sine of the inclination of the lower line to the axis of $y$ = the tangent of the inclination of the upper line to the axis of $x$. What is the distance of the point of contact from the origin when radius $= r$?

Let (fig. 12) $OX$, $OY$ be the co-ordinate axes, $BCA$ the circle, and $OA = d$ the quantity required.

Since $COY = (90 - COA)$

$\therefore$ $\sin COY = \cos COA = \dfrac{OA}{OC} = \dfrac{d}{\sqrt{(d^2+r^2)}}$ from the triangle $COA$,

and $\tan BOA = \dfrac{BA}{OA} = \dfrac{2r}{d}$;

$\therefore$ per question $\dfrac{d}{\sqrt{(d^2+r^2)}} = \dfrac{2r}{d}$,

$$\therefore d^4 = 4r^2(d^2+r^2),$$

$$\therefore d^4 - 4r^2d^2 + 4r^4 = 8r^4, \text{ completing square};$$

$$\therefore \text{ extracting } d^2 = r^2\{2 \pm \sqrt{8}\},$$

$$\therefore d = \pm r\sqrt{(2 \pm \sqrt{8})},$$

$\because \sqrt{8} > 2$, $\therefore \sqrt{(2-\sqrt{8})}$ is impossible: the negative sign under the radical must therefore be rejected, and then the distance of point of contact from origin is

$$d = \pm r\sqrt{\{2+\sqrt{8}\}},$$

the double sign shewing that two circles might be described to fulfil the given condition.

The co-ordinates of the extremities of a line, whose length is $l$, are $(a, o)$, $(o, b)$; and from the origin another line is drawn, cutting it in mean and extreme ratio. Required the co-ordinates of the point of intersection.

Let $OX$, $OY$, (fig. 13), be the co-ordinate axes.

Take $OA = a$, $OB = b$, and draw $BA$; then $BA = l$.

Let $OP$ be the line cutting $AB$ in mean and extreme ratio in the point $P$; draw $PN$ parallel to $OY$.

Let $PN = y$, $ON = x$, be the required co-ordinates of the point of intersection, and let $BP = z$.

Now since $BA$ is divided in mean and extreme ratio in the point $P$,

from Euclid (30, 6), $\dfrac{z}{l-z} = \dfrac{l-z}{l}$,

$$\therefore zl = l^2 - 2zl + z^2,$$

$$\therefore -l^2 = -3zl + z^2,$$

$$\therefore \frac{5}{4}l^2 = \frac{9}{4}l^2 - 3zl + z^2 \text{ completing square:}$$

and extracting,

$$\pm \frac{\sqrt{5}}{2} l = \frac{3}{2} l - z,$$

$$\therefore z = \frac{l}{2} . \{3 \mp \sqrt{5}\}.$$

Since a part must be less than the whole, the positive sign is inadmissible;

$$\therefore z = \frac{l}{2} . \{3 - \sqrt{5}\},$$

and
$$AP = l - z = \frac{l}{2} . \{\sqrt{5} - 1\}.$$

To find the co-ordinates of $P$, the similar triangles $ANP$, $AOB$ give

$$\frac{y}{b} = \frac{l - z}{l} = \frac{\sqrt{5} - 1}{2} \qquad \therefore y = \frac{b}{2} \{\sqrt{5} - 1\},$$

and
$$\frac{a}{x} = \frac{l}{z} = \frac{2}{3 - \sqrt{5}} \qquad \therefore x = \frac{a}{2} \{3 - \sqrt{5}\}.$$

A circle (radius $= r$) is described touching the positive axes. A chord is drawn cutting the vertical diameter in a point $pr$ from the axis of $x$, and is produced to the origin. Given that its inclination to the axis of $x$ is 30°, to determine the two parts into which it is divided by the vertical diameter, and the length of the part produced to origin.

Construct as in (fig. 14), then
$$\left.\begin{aligned} AP &= pr \\ PB &= (2 - p) r \\ OA &= r \end{aligned}\right\}$$

by question and construction; let $OR = z$, $RP = x$, $PQ = y$, the quantities to be determined.

By Trigonometry, $\sin 30° = \frac{1}{2}$, and because the chord is inclined at 30°,

$$\therefore \frac{PA}{PO} = \frac{1}{2} \text{ or } \frac{pr}{z + x} = \frac{1}{2},$$

$$\therefore 2pr = (z + x) \ldots\ldots (1),$$

also, because $RQ$ and $AB$ intersect within a circle,

$$\therefore \text{by (Euclid 35, 3)}, \quad AP . PB = RP . PQ,$$

$$\text{or} \quad p (2 - p) r^2 = xy \ldots\ldots\ldots (2);$$

also, because $OA$ touches the circle, and $ORPQ$ cuts it,

$\therefore$ by (Euclid 36, 3), $OR \,.\, RQ = OA^2$,

or $z(z + x + y) = r^2 \ldots\ldots (3)$;

(1), (2), (3), are sufficient to determine $x, y, z$.

Substituting (1) in (3) $z(2pr + y) = r^2$,

$$\therefore\ 2prz + yz = r^2 \ldots\ldots\ldots (4),$$

and from (2) $$xy = 2pr^2 - p^2r^2 \ldots\ldots\ldots\ldots (5),$$

(4) $\times\, x$ gives $2pxzx + yzx = r^2x$,

(5) $\times\, 2 \ldots\ldots\ldots\ldots xyz = 2pr^2z - p^2r^2z$;

$\therefore$ subtracting and transposing $(2prz - r^2)\,x = p^2r^2 - 2pr^2$,

which gives $$x = \frac{(2pr^2 - p^2r^2)\,z}{r^2 - 2prz},$$

substituting in (1) $$2pr = z + \frac{(2pr^2 - p^2r^2)\,z}{r^2 - 2prz},$$

from which quadratic $z$ can easily be ascertained in terms of $p$ and $r$. Let the value thus obtained be represented by $Z$, then substituting in (1), $$2pr - Z = x,$$

whence $x$ is determined; and substituting this value of $x$ in (2)

$$\frac{p\,(2 - p)\,r^2}{2pr - Z} = y,$$

and $x, y, z$, are all determined.

## PROBLEMS, &c.

1. The co-ordinates of the angular points of a triangle are $(o, o)$, $(a, o)$, $(o, a)$. Required its area, the radius of a circumscribed circle, and co-ordinates of the centre.

2. A circle (radius $= r$) is described touching the axis of $x$ in a point $mr$; from the origin two lines are drawn to the convex circumference, and if produced would pass respectively through the centre and extremity of the vertical diameter. What ratio have these lines to each other?

3. A circle is described touching the positive axes. The length of the shortest line that can be drawn to the circumference is $b$. What is the diameter?

4. A circle (radius $= r$) touches the axis of $x$, and the point of contact is taken for the origin. A chord $=$ radius is drawn from the origin, and a perpendicular from the centre upon this chord is produced to cut the axis of $x$. Determine in what point.

5. In the last problem determine the co-ordinates of the two points in which the line through the centre cuts the circumference.

6. A circle of given radius touches the positive axis of $y$; a line is drawn from the origin, passing through the centre, and terminated at the upper circumference, its length $= l$; a chord, whose length is $e$, is drawn perpendicular to this line. By how much must the chord be produced to cut the axis of $y$? Where does it cut it? And what are the co-ordinates of the centre of the circle?

7. Describe and find the area of the triangle, the co-ordinates of whose angular points are $(a, o)$, $(2a, o)$, $(3a, b)$; and what is the length of the perpendicular from $(2a, o)$ on the opposite side?

8. A circle touches the axis of $y$ in $b$: a tangent from the origin is inclined to the axis of $x$ at $30°$. Find the co-ordinates of point of contact and radius of circle.

9. A circle touches the positive axis of $y$; an equilateral triangle is placed within it, with one side parallel to the axis of $y$, the direction of another passing through the origin, $s$ being the length of a side: find the radius of the circle, and determine the point where it touches the axis of $y$.

10. The co-ordinates of the angular points of a triangle are $(a, b)$, $(c, d)$, $(e, f)$, where $b > d$ and $< f$, and $c > a$ and $< e$; the centre of the circumscribed circle lies over the point $(c, d)$, and $p$ is the length of the perpendicular from it on the side farthest from the axis of $y$. Required the radius and co-ordinates of the centre of the circle.

11. A circle (radius $= r$) touches the positive axes; the chord joining the points of contact is the base of an isosceles triangle, inscribed within the circle. Determine the sides of the triangle, the perpendicular on the base, and the co-ordinates of the vertex.

12. A circle (radius $= r$) is described touching the positive axis of $y$ and negative axis of $x$; from a point $(a, b)$ a line is drawn to the farther extremity of the horizontal diameter: what is the whole length of this line and of the part within the circle?

13. The co-ordinates of the angular points of a triangle are $(a, b)$, $(pa, 2b)$, $(o, 2b)$, $p$ being $> 1$, and the angle at $(a, b)$ being a right angle; required the area and the tangent of the angle at $(o, 2b)$.

14. A circle (radius $= r$) touches the positive axis of $x$, and from the extremity of the vertical diameter a line is drawn to the origin, the part of this line within the circle being of the same length as the radius; required the co-ordinates of the point in which it cuts the circle.

15. A circle (radius $= r$) is drawn touching the axis of $y$, and from the centre and extremity of the horizontal diameter two lines are drawn to the origin; given that the inclination of the line from the centre to the axis of $x$ equals the inclination of the other to the axis of $y$; to find the point of contact.

16. A circle (radius $= r$) is described touching the positive axes, and a line is drawn from the extremity of the vertical diameter to the origin; another line, parallel to the former, is drawn from the centre to meet the axis of $x$; compare the portions of these lines without the circle.

17. A circle is described touching the axis of $x$, and the point of contact is assumed as the origin, from which a chord is drawn inclined at 60° to the axis of $x$; from the extremity of a vertical diameter another chord is drawn to meet this at the circumference, and produced to cut the axis of $x$; determine in what point, and the whole length of this line.

18. A circle (radius $= r$) touches the positive axes, and from a point $(o, b)$ a line is drawn through the centre to meet the axis of $x$; find the co-ordinates of the two points in which it cuts the circle.

19. A circle touches the axis of $x$, and the point of contact is the origin; from $(-c, o)$ a line equal in length to the diameter is drawn meeting the concave circumference in $(a, b)$: what are

the co-ordinates of the point in which this line cuts the convex circumference, and what is the radius?

20. A circle touches the axis of $y$ in $b$, and from this point a chord whose length is $c$ is drawn; if a line be drawn from the other extremity of $c$ to the origin it will be a tangent to the circle. Required the co-ordinates of that extremity.

21. Find the radius in the last case.

22. A circle touches the axis of $y$ in $(b)$; in it an equilateral triangle is placed with one side $s$ parallel to the axis of $y$; required the point in which this side produced cuts the axis of $x$. Also the points in which the other sides produced cut the axis of $y$, and the radius of the circle.

23. The co-ordinates of the centre a of circle are $(a, b)$, and within it is described an isosceles triangle, the base of which is vertical; the lowest inclined side being produced cuts the axis of $y$ in a point from which an horizontal line is a tangent to the circle; determine this point and the sides of the triangle, the radius of the circle being $r$.

24. A circle touches the positive axes, and from the origin a line is drawn to the farthest point in the circumference, the length of this line is $l$. What are the co-ordinates of its extremity? This point being taken as the vertex, an isosceles triangle is described within the circle, and the co-ordinates of the other angular points are $(a, b)$ $(b, a)$ respectively. Determine the sides of the triangle and the radius of the circle.

25. A circle touches the positive axes, and from the point of contact in the axis of $y$ a chord whose length $(c)$ is drawn inclined at an angle $\left(\tan^{-1}\frac{b}{a}\right)$. What are the co-ordinates of the point in which it cuts the circle, and what is the radius?

26. The co-ordinates of the angular points of a triangle are $(a, o)$ $(a, b)$ and its area $= A$; required the co-ordinates of the third angular point, the perpendicular from $(a, b)$ on the opposite side being $p$. Explain the double sign.

27. A circle (radius $= r$) is described touching the positive axes; a line whose length $mr$ is drawn perpendicular to the axis

of $x$, and terminated by the upper circumference of the circle, the length of the part within the circle being $r$; determine the point in the axis of $x$ from which it is drawn.

28. In a circle described as in (27), three chords are drawn, and being produced pass through the origin; the lengths of the chords and parts produced are respectively $mr, (m+1)r, (m+2)r$; what ratio have the chords to each other?

29. Find the areas of the triangles, the co-ordinates of whose angular points are $(o, d)$ $(c, o)$ $(a, b)$ and $(a, b)$ $(c, b)$ $(e, f)$.

30. A circle is described touching the axis of $x$, and the point of contact is taken for the origin; from that point a chord (length $c$) is drawn inclined at 60° to the axis of $x$. From the extremity of the vertical diameter another chord is drawn to meet this at the circumference, and produced to cut the axis of $x$; determine in what point, and what is the area of the circle.

31. A circle touches the axis of $y$ in $b$; a line which is a geometrical mean between $b$ and the radius is drawn parallel to the axis of $x$ from such a point in the axis of $y$, that the part within the circle is of the same length as the radius; find the co-ordinates of the points in which it cuts the circle, and the circumference of the circle.

32. $d$ is the diameter of a circle touching the axis of $x$ at the origin; from $(a, b)$ a line is drawn to meet the axis of $x$, and also the vertical diameter in a point $\frac{r}{n}$ from the upper extremity, $r$ being the radius; find where it meets the axis of $x$, and the co-ordinates of the point in which it intersects the circle.

33. A circle touches the axis of $y$ in $c$, and if a tangent be drawn to the circle at a point $(a, b)$ it will pass through the origin; find the length of the chord joining the points $c$ and $(a, b)$, and the distance of its middle point from the origin.

34. An equilateral triangle is placed in a circle with one side parallel to the axis of $y$, the direction of the lowest inclined side passing through the origin, the distance of the nearest angular point from the origin is $d$; find the area of the circle and the co-ordinates of its centre. Side of the triangle $= s$.

35. A circle of given radius touches the positive axes, and from the point of contact in the axis of $y$ a chord is drawn; what is the length of this chord, and what the co-ordinates of the point in which it meets the circle when a parallel line from the origin passes through the extremity of the horizontal diameter?

36. The co-ordinates of the angular points of a quadrilateral figure are $(a, o)\,(a, b)\,(2a, o)(c, d)$ where $c > a$ $d > b$, its area $= A$; find the length of the perpendiculars from the nearest and farthest angular points upon the diagonal passing through the other two.

37. $(-a, o)$ $(o, o)$ $(a, b)$ being the co-ordinates of the angular points of a triangle, find its area; also find the area and sides of the triangle whose position is determined by the co-ordinates $(-a, -b)$ $(a, -2b)$ $(2a, o)$.

38. A circle touches the positive axes; from the origin three chords are drawn inclined at given angles $\alpha$ $\beta$ $\gamma$, and perpendiculars from their extremities cut the axis of $x$ in $a$, $b$, $c$ respectively; determine the points in which the chords cut a vertical diameter.

39. A circle is described touching the axis of $x$ at the origin; from a point $d$ in the axis of $x$ a line is drawn passing through $(a, b)$ in the circumference of the circle, and cutting the vertical diameter and the circle in another point. The part of this line within the circle is divided by the diameter in parts which are as $2:3$; required co-ordinates of the second point in which it cuts the circle, the area of the circle, and the two parts in which the diameter is divided.

40. A circle touches the positive axis of $x$; from the origin a line (length $l$) is drawn inclined at an angle $\alpha$, and just meets the circle; being produced it cuts the diameter whose length is $D$; determine in what point.

41. The area of a circle is $A$, that of an isosceles triangle within it $B$, and the co-ordinates of two of its angular points are $(r, o)$ and $(o, r)$, $r$ being the radius; determine the radius and sides of the triangle.

42. A circle touches the axis of $y$ in $b$; from the extremity of

the horizontal diameter a line is drawn to the axis of $y$, which it meets in $c$: this line, which is bisected by the vertical diameter, divides that diameter into two parts, having the ratio $1:3$; find the length of the line, and radius of the circle.

43. Find the area of a quadrilateral the co-ordinates of whose angular points are $(o, o)$ $(a, b)$ $(3a, b)$ $(3a, 2b)$.

44. An equilateral triangle is placed in a circle with one side perpendicular to the axis of $x$; the co-ordinates of the nearest angular point are $(a, b)$, and the inclined side nearest the axis of $x$ being produced cuts it in $(c)$; from this point a line (length $l$) is drawn touching the circle at its extremity: find the areas of the triangle and circle, and co-ordinates of the centre.

45. A circle touches the axis of $y$ in $b$, and a triangle is described within it, the co-ordinates of whose angular points are $(o, b)$ $(a, c)$ $(a, 2c)$; determine the sides and the radius of the circle.

46. $(a, b)$ are the co-ordinates of the centre of a circle in which is placed an equilateral triangle, whose vertex coincides with the extremity of a horizontal diameter, the nearest angular point is distant $d$ from the origin; find the radius of the circle, and sides of the triangle.

47. A circle is described touching the positive axes; two equal chords (length $c$) are placed in the circle, one inclined and the other perpendicular to the axis of $x$, and produced to cut the axis of $x$ in $a$, $d$, respectively; by how much is each chord produced?

48. The co-ordinates of the centre of a circle are $(o, d)$, a chord is drawn the co-ordinates of whose extremities are $(-a, b)$ $(e, f)$; what is the circumference of the circle, and in what points does it cut the axis of $y$?

49. The circumference of a circle is $C$, and the perimeter of an inscribed equilateral triangle $P$; what are the areas of the circle and triangle?

50. From a point $b$ in a vertical diameter, a line (length $c$) is drawn to the circumference, and is inclined at $a$ to the axis of $x$; what is the area of the circle?

51. *Hippocrates' Theorem.* $ABC$ (fig. 15) is a right-angled isosceles triangle, on the sides of which, as diameters, the semicircles $AaBcC$, $ADB$, $BEC$, are described; prove that the area of the triangle $ABC$ is equal to the sum of the areas of the lunulæ $ADBaA$, $BECcB$.

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In the introductory observations to this chapter, the quantities $x$ and $y$ were assumed to represent indeterminate distances only, and were affected with the positive or negative sign, according to the direction in which the point to which they referred was assumed to lie, so that $(x, y)$ would represent any point in the first quadrant: at the same time it was said, that the determination of the values of $x$ and $y$ was to be the object of a problem.

Now if a problem were to give the following values,

$$x = a - c, \quad y = b,$$

the position of the point $(x, y)$ would clearly depend upon the relative values of $a$ and $c$.

If from some data it resulted that $c = a$, then $(o, b)$ would be the point required: that is, it would be in the positive axis of $y$.

If $c$ were $> a$ then $a - c$. would be negative, and the required point would be $[-(c-a), b]$; that is, it would be in the second quadrant: and by making similar suppositions for $y$, the student will easily understand why a point represented hypothetically by $(x, y)$, and therefore tacitly assumed as situated in the positive quadrant, may, when the problem is solved, be found not to be in that quadrant at all.

To proceed correctly then, it appears that this tacit assumption must be removed, or at least that the signification of the expression $(x, y)$ must be so far extended, as to admit that it may refer to any point in any one of the quadrants or axes; so that an equation in $x$ and $y$ obtained by expressing that the point $(x, y)$ fulfils certain conditions of a problem, shall hold not for a point in the positive quadrant only, but for points in any of the other quadrants. And thus, when as in the next chapter, the motion of a point is described, and, for convenience of investigation, a figure is drawn in the positive quadrant; the equation in $x$ and $y$ thence obtained will with as great propriety refer to the motion in the 2nd, 3rd, and 4th quadrants, if the data do not limit or restrain the motion within any particular one. If the data do limit the motion, the equation will shew it. See Locus (25), p. 24, where the form of the equation excludes the negative values of $x$.

# CHAPTER II.

## 4. *Locus, Equation to a Locus, &c.*

A *Line*, whether straight or curved, may be conceived generated by a *moving point*, and the position of a point in a plane is determined by means of its co-ordinates, rectangular, oblique, or polar. If then at any time we know the co-ordinates of the moving point, we know the position of *one* point in *the line* which it generates, and might easily represent it geometrically: and if we knew the position of all the points in the line by the same process that we represent one, we might *represent all geometrically*, i.e. we might *trace the line.*

Now the co-ordinates of a point (when rectangular or oblique) are two quantities of the same kind (two distances), and admit of being compared; the ordinate therefore always bears some proportion to the abscissa, excepting of course when either of them = 0. When this proportion is known, *the relation between the co-ordinates* of a point is said to be known, and may be *expressed in an Equation*, which is then called the *equation to the point.*

Let a point be conceived to move *continually in the same manner*, i.e. the *same relation always subsisting between its co-ordinates*. This relation is called *the Law of the point's motion*, by which the position of every point in the line or curve generated is determined. When then the law is given, we may, taking $x$ and $y$ to represent the co-ordinates of any point, obtain an *equation between $x$ and $y$*, which being true for *every point** of the curve or line, enables us to trace the same, and is for that reason called *the Equation to the curve or line*, or its algebraical representation; while the curve or line which, when traced, geometrically represents the *path* of the moving point, is called the *Locus of the Equation.*

* For by assumption it is the Equation to *any point.* See Note (p. 20).

Suppose, for instance, a flat ruler laid upon this sheet of paper, and a straight line to be drawn with a pencil moving along it. In this case, the *moving point* would be the *extremity of the pencil*, the *law of its motion; always to touch* the ruler and paper, and *the line* itself, would be the *locus* of the equation expressing that law.

Again, suppose an inextensible string attached to the pencil, and one end of the string to be fixed; the pencil held vertically, and touching the paper, and moved so as always to keep the string stretched. The law of motion would in this case be, that the moving point (the extremity of the pencil) be always at the same distance from a fixed point (the fixed end of the string) and, as the student knows, a circle would be traced. Consequently the locus of an equation expressing that a point moves so as to be always at the same distance from a fixed point, is a circle.

These two simple cases have been adduced to fix the student's ideas with respect to the terms, locus, &c. It will occur to him, that since the geometrical representation of a circle differs essentially from that of a straight line, the *algebraical representation* of the one, ought by some peculiarity to be distinguished from the other. This is actually the case: the forms of the equations, as the student will see in the sequel, are very different.

That this should be the case, he might perhaps infer from the fact, that the laws of the describing point's motion are different: but this does not hold generally, as he will soon understand. In the case of the circle for example: *one* way to find its equation is to express that the distance of the centre from any point in the circumference is constant; but this is only one of several properties which the student knows to belong to the circle. He is aware that if two lines be drawn from the extremities of a diameter through any point in the circumference, these lines cut each other at a right angle; and that the angle in a segment of a circle is constant. Now these properties (not to mention others) lead to laws different from the first, yet the resulting equation is in every case of the same form: from which it follows, that points moving according to different laws may nevertheless describe similar curves.

As the object of the present chapter is merely to present to the student some problems on Loci, as farther practice in the application of Algebra to Geometry, nothing need at present be said respecting the forms of the equations he will obtain. A few hints as to the method of arriving at them is all that is necessary.

(1) The locus or curve is supposed referred to rectangular axes.

(2) The equation to a plane locus, so referred, being an equation in $x$ and $y$ only, no other *variable* quantities must be found in it.

If, therefore, to express a given condition, it should be necessary to introduce any *functions of the angle* (say $\theta$) which the radius vector (the line drawn from the moving point to the origin) makes *with the axis of $x$ or $y$*; as $\tan\theta$, $\sin\theta$, $\cos\theta$, &c. these *must be eliminated* by substituting their equivalents in $x$ and $y$, as $\frac{y}{x}$ $\frac{y}{\sqrt{(x^2+y^2)}}$ $\frac{x}{\sqrt{(x^2+y^2)}}$ &c. as is clear on constructing for (A).

(3) In all cases where the moving point is connected with rods and strings, a figure such as the question suggests should be drawn, and as accurately as possible. When in the enunciation the words *vertical* and *horizontal* are used, they always signify *directions parallel to the axis of* y, and *axis of* x respectively.

(4) The figure should represent the system in a *general position*, that is, in such a position that the *radius vector coincides not with either of the axes.*

(5) Make no suppositions but such as are allowed by the question: for example, it would be inconsistent with (4) to assume that the radius vector is inclined at any particular angle to either of the co-ordinate axes, and thence deduce a relation between the co-ordinates or any given lines.

These observations being borne in mind, the following two steps will be sufficient.

Having sketched the figure which exhibits the position of the moving point, let a line be drawn from it to the origin, and another perpendicular axis of $x$: these are respectively the radius vector ($\rho$), and ordinate ($y$), and form with the abscissa ($x$) a right-angled triangle, whence the relation

$$\rho^2 = x^2 + y^2 \ldots\ldots\ldots\ldots\ldots (A),$$

which always holds independently of the data, when a locus is referred to rectangular co-ordinate axes.

If the question give a relation between the radius vector ($\rho$) and constant quantities, or between $\rho$ constant quantities, and one or both of the co-ordinates, find an expression for $\rho^2$ by means

of this relation, and substitute in (A). The resulting equation, which should be simplified as much as possible by taking away quantities similarly involved on each side, is the required equation to the locus.

But if the question do not give such a relation, the student must discover, from the figure, a relation between $\rho$ and other (given) lines of the system by means of triangle. A little practice will soon enable him to discover which will best answer his purpose; and having determined on the triangle, he knows the trigonometrical relations, viz. that the sides are proportional to the sines of the opposite angles, and that one side can always be expressed in terms of the other two and included angle. He is thus enabled to find an expression for $\rho^2$ which is to be substituted in (A).

These observations shall be applied to loci (25), (24), (61), and (22).

Locus (25). Take $OX$, $OY$ (fig. 16) for the co-ordinate axes, and let $SQA$ be the circle which the extremity $Q$ of the rod $OQ$ would describe by revolving about $O$.

$SQ$ the length of string drawn up, and $P$ its middle point, the equation to whose locus is required, draw the dotted lines $OP$, $PN$.

Now because $SQ$ is a chord of a circle $SQA$, and $OP$ is drawn from the centre to the middle point, $OP$ is perpendicular to $SQ$, and therefore the triangles $PON$, $PSO$ are similar. By means of the latter, the radius vector is connected with $OS = OQ = a$, the given constant quantity. This is enough.

Let $OP = \rho$, $PN = y$, $ON = x$, $PON = \theta$, $SQ = s$, which are all variable, and $OQ = a$, per question;

from the triangle $PON$, $\rho^2 = x^2 + y^2$ ................ (1),

. . . $POS$, $\rho^2 = a^2 - \left(\frac{s}{2}\right)^2$ .............. (2),

. . . $POS$, $\dfrac{\frac{s}{2}}{a} = \sin\theta = \dfrac{y}{\sqrt{(x^2 + y^2)}}$

$$\therefore \frac{s}{2} = \frac{ay}{\sqrt{(x^2 + y^2)}} \text{ .................. (3).}$$

Substituting from (1) and (3) into (2),

$$x^2 + y^2 = a^2 - \frac{a^2y^2}{x^2+y^2},$$

$$\therefore\ (x^2+y^2)^2 = a^2x^2,$$

extracting $$x^2 + y^2 = \pm\, ax;$$

because the sum of two squares cannot be negative, the negative sign must be rejected. Hence the equation to the locus is

$$x^2 + y^2 = ax,$$

which may be put under the form

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}.$$

LOCUS (24). Let $OX$, $OY$ (fig. 17) be the co-ordinate axes, and $OPEL$ the revolving variable line, $QPp$ the line perpendicular to the axis of $y$, which divides $OEL$ in $P$ as per question, then $P$ is the point whose locus is required. Draw $PN$ and $EM$ both perpendicular to the axis of $x$.

Now the relation of $OP$ to $OL$ is known from question, also that of $EL$ to $EM$, and $EM$ is connected with $OE$ the given constant quantity, which is all that is required.

Let $OP = \rho$, $PN = y$, $ON = x$, $OEL = l$, $PON = \theta$, which are all variable, and $OE = a$;

$$l = OE + EL = a + pEM \text{ per question.}\ \ldots(1),$$

and $$\frac{\rho}{l} = \frac{m}{n} \qquad \ldots\ldots(2);$$

from triangle $EOM$, $\dfrac{EM}{a} = \sin\theta$,

$$\therefore\ EM = \frac{ay}{\sqrt{(x^2+y^2)}} \ldots\ldots\ldots\ldots (3);$$

from triangle $PON$, $$\rho^2 = x^2 + y^2 \ldots\ldots\ldots\ldots (4),$$

$$\therefore\ x^2 + y^2 = \frac{m^2}{n^2}(l^2), \text{ from } (2) \ldots\ldots (5).$$

Substituting (3) in (1), and (1) in (5),

$$x^2 + y^2 = \frac{m^2}{n^2}\left\{a + \frac{pay}{\sqrt{(x^2+y^2)}}\right\}^2,$$

which is the equation to the locus of $P$.

LOCUS (61). Let (fig. 76) the given diameter be taken for the

axis of $x$, and the centre of the circle for the origin, the locus being referred to rectangular axes $OX$, $OY$, through that point.

Let $PTR$, $PtQ$ be the tangents to the circle, cutting the axis of $x$ in $R$ and $Q$ respectively; draw $OP$, $PN$ perpendicular to $OX$, and the radii $OT$, $Ot$; then $OP=\rho$, $PN=y$, $ON=x$.

Let $OR=z_1$, $OQ=z_2$; $c^2=$ rectangle under $OR$, $OQ$.

First, $$\rho^2=x^2+y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and by question $$z_1z_2=c^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

If then the values of $z_1$, $z_2$, be found in terms of $\rho$, and its equivalent substituted from (1), the result substituted in (2) will be an equation between the given constant, $x$, and $y$.

From triangle $POR$, $$\frac{z_1}{\rho}=\frac{\sin OPR}{\sin PRO},$$

$$\therefore\ z_1=\rho\,.\,\frac{\sin OPR}{\sin PRO};$$

and similarly for $z_2$ from triangle $POQ$,

$$z_2=\rho\,.\,\frac{\sin OPQ}{\sin PQO}.$$

It now only remains to find expressions for the sines without introducing any other than constant quantities, and $x$, $y$.

The fact that the radius of a circle is perpendicular to the line touching the circle, renders this easy; for,

from the triangle $POT$, $\sin OPR=\sin OPT=\dfrac{r}{\rho}=\cos POT$,

from the triangle $TOR$, $\sin PRO=\sin TRO=\cos TOR$,

$$=\cos(POR-POT).$$

By Trigonometry,

$$\cos(POR-POT)=\cos POR\cos POT+\sin POR\sin POT;$$

now $$\cos POR=\cos PON=\frac{x}{\rho},$$

$$\sin POR=\sin PON=\frac{y}{\rho},$$

$$\sin POT=\sqrt{(1-\cos^2 POT)}=\frac{\sqrt{(\rho^2-r^2)}}{\rho};$$

substituting these values in the above expression,

$$\sin PRO=\frac{x}{\rho}\,.\,\frac{r}{\rho}+\frac{y}{\rho}\,.\,\frac{\sqrt{(\rho^2-r^2)}}{\rho}.$$

Hence $$z_1 = \frac{r\rho^2}{\sqrt{(\rho^2 - r^2)} \cdot y + xr}.$$

Again, from the triangle $POt$, $\sin OPQ = \sin OPt = \frac{r}{\rho}$,

from the triangle $QtO$, $\sin PQO = \sin tQO = \cos tOQ$
$= \cos (POQ - POt)$,

and $\cos (POQ - POt) = \cos POQ \cos POt + \sin POQ \sin POt$.

Now the angle $POQ = (\pi - PON)$,

and by Trigonometry, $\cos (\pi - PON) = - \cos PON$,
$\sin (\pi - PON) = \sin PON$,

and since $\sin POt = \sqrt{(1 - \cos^2 POt)} = \frac{\sqrt{(\rho^2 - r^2)}}{\rho}$.

By substitution,

$$\sin POQ = \frac{\sqrt{(\rho^2 - r^2)}}{\rho} \cdot \frac{y}{\rho} - \frac{x}{\rho} \cdot \frac{r}{\rho},$$

$$\therefore z_2 = \frac{r\rho^2}{\sqrt{(\rho^2 - r^2)} \cdot y - xr};$$

and substituting in (2),

$$c^2 = \frac{r^2\rho^4}{(\rho^2 - r^2) y^2 - x^2r^2},$$

$$= \frac{r^2\rho^4}{\rho^2y^2 - r^2 (x^2 + y^2)}.$$

Substituting for $\rho^2$ and reducing,

$$c^2 = \frac{r^2(x^2 + y^2)}{y^2 - r^2},$$

$$\therefore y^2(c^2 - r^2) = r^2(c^2 + x^2),$$

or $$y^2 = \frac{r^2}{c^2 - r^2} (c^2 + x^2),$$

which is the equation to the locus of $P$.

Locus (22). Let $OX$, $OY$ (fig. 18) be the co-ordinate axes; $OH$, $HP$ the two rods, $H$ being the hinge and $P$ the point whose locus is required.

Draw $OP$, also $PN$ perpendicular to the axis of $x$,
$Hh$ . . . . . .
$HK$ parallel . . . . .

then $HK = hN \because HKNh$ is a parallelogram
$= ON - Oh$
$= x - Oh \because ON = x$.

Let $OP = \rho$, $PN = y$, $HOh = \phi$, all variable,
then $PHl = \phi$, and $PHK = 2\phi$ per question,
$OH = a$, $HP = b$, with which the radius vector is connected by means of the triangle $POH$.

from the triangle $PON$ $\quad \rho^2 = x^2 + y^2 \quad \ldots\ldots\ldots\ldots\ldots (1),$

$$\begin{aligned} \ldots \quad POH \quad \rho^2 &= a^2 + b^2 - 2ab \cos PHO \\ &= a^2 + b^2 - 2ab \cos(\pi - \phi) \\ &= a^2 + b^2 + 2ab \cos \phi \quad \ldots\ldots (2), \end{aligned}$$

Substituting from (1) in (2) and transposing,

$$\cos \phi = \frac{x^2 + y^2 - (a^2 + b^2)}{2ab} \quad \ldots\ldots\ldots (3),$$

from the triangle $PHK$, $\cos 2\phi = \dfrac{HK}{b} = \dfrac{x - Oh}{b} \ldots\ldots\ldots (4),$

$$\ldots \quad HOh, \quad \frac{Oh}{a} = \cos \phi \therefore Oh = a \cos \phi \ldots (5),$$

substituting (5) in (4) $\cos 2\phi = \dfrac{x - a \cos \phi}{b} \ldots\ldots\ldots\ldots (6);$

but from Trigonometry $\cos 2\phi = 2 \cos^2 \phi - 1 \ldots\ldots\ldots\ldots (7),$
substituting (7) in (6) and transposing,

$$\left(2 \cos \phi + \frac{a}{b}\right) \cos \phi = \frac{x + b}{b} \ldots\ldots\ldots\ldots (8),$$

and substituting (3) in (8) to eliminate $\phi$,

$$\left(\frac{x^2 + y^2 - (a^2 + b^2)}{ab} + \frac{a}{b}\right) . \frac{x^2 + y^2 - (a^2 + b^2)}{2ab} = \frac{x + b}{b},$$

and reducing

$$(x^2 + y^2 - b^2) \left\{(x^2 + y^2 - b^2) - a^2\right\} = 2a^2 b\,(x + b) \ldots (9),$$

which is the equation to the locus.

This equation, as has been before observed, expresses the relation subsisting between the constants $a$, $b$, and the co-ordinates of *every point* of the curve or locus in which the point $P$ moves. When, therefore, it is required to find the relation which obtains when the system is in a *particular position*, nothing more is necessary than to substitute in (9) those values of the variables and constants which determine that position; for example, required the relation which obtains when $P$ meets the axis of $y$.

Its co-ordinates being then $(o, y)$ make $x = o$ in (9), and the result is

$$(y^2 - b^2)\{(y^2 - b^2) - a^2\} = 2a^2b^2,$$

whence $$y^4 - y^2(2b^2 + a^2) = a^2b^2 - b^4,$$

completing square,

$$y^4 - y^2(2b^2 + a^2) + \frac{(2b^2 + a^2)^2}{4} = a^2b^2 - b^4 + \frac{(2b^2 + a^2)^2}{4}$$
$$= \frac{a^2}{4}(a^2 + 8b^2);$$

extracting $$y^2 - \frac{2b^2 + a^2}{2} = \pm \frac{a}{2}\sqrt{(a^2 + 8b^2)}.$$

If the quantities on the left-hand side had been connected by the positive, the negative sign on the right-hand side would have to be rejected for the reason assigned in investigating locus (25). But because the difference of two or more squares may lead to a negative result, the double sign must be used.

Transposing and extracting again,

$$y = \pm \frac{1}{\sqrt{2}}\sqrt{\{(2b^2 + a^2) \pm a\sqrt{(a^2 + 8b^2)}\}} \;\ldots\ldots (10).$$

This result, which gives the values of $y$, when $P$ meets the axis of $y$, that is, determines the points in which $P$ cuts the axis of $y$, it will be useful to investigate fully.

First, suppose the rods equal; *i. e.* suppose $b = a$ (10) becomes

$$y = \pm \frac{1}{\sqrt{2}}\sqrt{(3a^2 \pm 3a^2)} \;\ldots\ldots\ldots\ldots (11);$$

if the positive sign be used under the radical

$$y = \pm \frac{1}{\sqrt{2}}\sqrt{(6a^2)}$$
$$= \pm\sqrt{3}\,.\,a,$$

which shews that the point $P$, when the rods are equal, meets the axis of $y$ in two points equally distant from the origin, one above and the other below the axis of $x$.

If the negative sign be used under the radical

$$y = o,$$

a result, of course, not susceptible of the double sign; it shews that the point $P$ meets the axis of $y$ at the origin, which is also correct on the hypothesis of the rods being equal. For, when

$OH$ has revolved through 180° about $O$, and $HP$ through the same angle about $H$, the point $P$ must manifestly be at $O$.

Secondly, suppose $b \sim a$, or the rods unequal; then, because the square root of a negative quantity is impossible, the double sign under the radical cannot be used without this condition, that the expression under the radical remain positive, that is,

$$2b^2 + a^2 \text{ must be} > a\sqrt{(a^2 + 8b^2)},$$

or
$$(2b^2 + a^2)^2 > a^2(a^2 + 8b^2),$$

therefore
$$4b^4 + 4a^2b^2 + a^4 > a^4 + 8a^2b^2,$$

$$\therefore b > a;$$

when this is the case $y$ has 4 values (2 pairs with opposite signs). That is, if the rod $HP$ be > $OH$, the point $P$ will meet the axis of $y$ in four points, two above and two below the axis of $x$.

If $b < a$, analytical considerations have just shewn that the negative sign is inadmissible, in other words, that $P$ *then* meets the axis of $y$ in *two* points *only*. That this result is the *only consistent* one with the geometry of the system, will soon appear evident. For, as $OH$ moves from the horizontal to the vertical position, $P$ must *first* come to the axis of $y$, then pass beyond it. When $OH$ has revolved through 180°, and coincides with the negative axis of $x$, $HP$, which has revolved through the same angle about $H$, also coincides with that axis, but being < $OP$, cannot meet the axis of $y$. *After OH has coincided* with the negative axis of $y$, $P$ *meets* that axis, then passes beyond it; and when $OH$ comes into its initial position, after revolving through 360°, $HP$ also comes into its initial position, and however long the motion is continued, $P$ can only retrace the same curve, therefore *cannot cut* the axis of $y$, *except in the two points* already noticed. By a similar investigation the propriety of the four values of $y$, when $b > a$, will become apparent.

This proposition, locus (22), the geometrical conditions of which are immediately obtained from the analytical expression, has been investigated at some length, in order to impress on the student the necessity of *attending to the double sign*, whenever and how often soever it occurs, as well as to exhibit in a small way the powers of analysis when applied to Geometry. It has

been shewn that, according as the upper is less, equal, or greater than the lower rod, its free extremity, moving according to the given law, touches the axis of $y$ in two, three, or four different points, making in all nine different results, every one of which is included and easily deducible from (9) or (10).

There are two others. Taking the positive sign in (11)

$$y = \sqrt{3} \, . \, a,$$

therefore, dividing by 2, $$\frac{\frac{y}{2}}{a} = \frac{\sqrt{3}}{2},$$

by Trigonometry $$\frac{\sqrt{3}}{2} = \sin 60°,$$

$$\therefore \; \frac{\frac{y}{2}}{a} = \sin 60°,$$

which expresses that the lower of the equal rods, when $P$ meets the positive axis of $y$, is inclined to the axis of $x$ at 60°.

Taking the negative sign in (11), it may similarly be shewn that the lower rod is inclined at $(\pi + 60)$ to the axis of $x$, when $P$ meets the negative axis of $y$. Both these results, on constructing and investigating, the student will find to be correct.

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## LOCI, &c.

In the following enunciations, find the locus means the same as find the equation to the locus.

1. A string carrying a weight so that the string is always perpendicular to the axis of $x$, hangs from a revolving rod, the lengths of string and rod varying so that the length of the rod equals $m$ times the length of the string, and the length of the string is a mean proportional between the co-ordinates of the weight. Required the equation to the locus of the weight.

2. A rod (length $d$) carries at one extremity another rod (length $a$), which always remains horizontal, so that when the rod $d$ is vertical the two form a T; find the locus of either extremity of rod $a$, as rod $d$ revolves about its lower end.

3. At a distance $(-d)$ conceive a hole in the axis of $x$, in which suppose a rod to be introduced and made to revolve, the length at the same time increasing so as to bear a constant ratio $n$ to the distance of its extremity from the origin; find the locus of extremity.

4. There are two points, one of which is fixed at the origin, the other moveable, but distant $(a)$ from the fixed point at the beginning of its motion. It moves so that at any time the square of its distance from origin, *minus* twice the square of ordinate, is equal to the square of its original distance multiplied by the cosine of the inclination of its radius vector to the axis of $x$.

5. A rod (length $a$) is moved in such a manner that one extremity always touches the axis of $x$ while the other revolves, and a ring moves along the rod so that its distance from the origin is always equal the length of the rod + or − the difference of its co-ordinates; find the locus of the ring.

6. Suppose a ruler (length $l$) to revolve about one extremity, and a pencil to move along the ruler in such a manner that at any time the sum of its distance from point of revolution and abscissa of revolving extremity may equal $p$ times the square of the pencil's ordinate divided by their difference; find the locus traced by pencil.

7. A rod passing through the origin is inclined at $\alpha$ to the axis of $x$; a string carrying a weight slides down the rod by means of a ring, and during its descent varies its length so as to be always $n$ times the distance of weight from axis of $x$. Required locus of weight.

8. A line revolving about the origin intersects another revolving about a point in a vertical line drawn through $+a$ in axis of $x$. The motions are so combined that the distance of the point of intersection from the vertical line is always $n$ times its distance from the axis of $x$. Required the equation to its locus.

9. A beam (length $l$) touches the axes of $y$ and $x$, the former in a point $a$; it is left to slide, and a point moves up the beam as fast as the extremity in contact with the axis of $y$ slides down that axis; find the locus of the point.

10. It is obvious that in (9) the point moving up the beam cannot rise above a certain height from the axis of $x$. Determine this greatest height, the distance it has run up the beam, and how far it is from the origin.

11. A line revolves about a point $(a, o)$ from the vertical position towards the axis of $y$; another line coinciding at first with the axis of $x$, revolves about the origin towards the axis of $y$, and makes always the same angle with the axis of $x$ as the first does with the vertical; find the locus of intersection.

12. A boy flying a kite sends up a messenger; find the locus of the messenger, supposing the boy the origin, the string (length $s$) always straight, and the height of the kite above ground always equal the distance of the messenger from the boy.

13. The messenger comes down again, and its distance from the axis of $y$ always equals the kite's distance from the axis of $x$; find the locus of the messenger.

14. Let $b$, $a$ be the base and perpendicular altitude of a right-angled triangle, the angular point being the origin, and the axis of $y$ parallel to $a$; and suppose the hypothenuse to be moved so that one extremity may always touch the axis of $y$, while in the direction of its length the hypothenuse slides on the extremity of the perpendicular: find the locus of the other extremity.

15. Two rods, $a$, $b$, are joined together so as to include an angle $(\pi - a)$ and at the extremity of $b$; a string $(s)$ is attached which carries a weight: find the locus of the weight as the system revolves about the extremity of $a$.

16. Supposing the string of such length that the weight is always below the axis of $x$, find the locus of the intersection of the string and rod $a$.

17. A pen, the length of which is $p$, is held vertically, the nib touching the paper, and moved from left to right; as the nib moves from its first position through any distance, the feathered end approaches the paper by the same distance; find the equation to the curve described by the extremity of the feather.

18. A line rests in an inclined position, meeting the axes of $x$ and $y$, and then slides down; find the locus of a point which divides the whole line into two parts, $a$, $b$.

19. Find the locus of the middle point from the last result.

20. A line $ABC$ revolves about $A$; $AB$ is constant and equals $a$; $BC$ is variable, being always equal to $n$ times the height of $B$ above the axis of $x$; find the equation to the locus of $C$.

21. A pair of scissors being placed upon the table with the pointed end upwards, one of the lower ends is pressed firmly against a point on the table, so as to allow that branch to have a motion of rotation about it, while the other branch moves freely; find the locus of the pointed extremity of the free branch, both branches are divided by the connecting screw in parts $a$, $c$, $c$ being greater than $a$.

22. Two rods, $a$, $b$, are connected by a string or hinge, so that two extremities are in contact. The other extremity of rod $a$ is fixed to a point in the axis of $x$, about which it revolves, while rod $b$ revolves with the same angular velocity about the hinge. Required locus of the other extremity of $b$.

23. As a point describes the circumference of a circle, chords are drawn from it to the extremity of a diameter, and a line parallel to the diameter moves so as always to intersect these chords in their middle points; find the locus of either intersection.

24. A line $OEL$ revolves about $A$, varying its length continually. $OE$ is constant, and $EL$ increases by $n$ times the height of $E$ above the axis of $x$. As $OEL$ revolves, a line perpendicular to the axis of $y$ moves so as always to divide the whole increased line in a point, the distance of which from the origin has to the whole line the ratio $m : n$; find locus.

25. A rod (length $a$), one extremity of which is at the origin, is laid so as to coincide with the axis of $x$: at the other extremity is suspended a string carrying a weight, the string passing through a hole in the axis of $x$ at a distance $a$ from the origin. As the rod revolves, the string is pulled up and the weight raised; find the locus of the middle point of the portion of string above the axis of $x$.

26. The length of the string being $l$, find the distance of the weight from the axis of $x$, when the rod has revolved through $60°$.

27. Find the locus of the middle point of the string when the hole in the axis of $x$ is at a distance $b$ from the origin, and $b$ either $>$ or $< a$; find also the elevation of the weight above its primitive position, corresponding to a revolution through an angle $\beta$.

28. Conceive the axis of $y$ to be a groove, in which runs the extremity of a line {length $(l+c)$}; at a distance $d$ in axis of $x$ is a fixed point about which revolves a line length $h$, the other extremity of which is joined by means of a pin to the line $(l+c)$, meeting it in $l$ from the extremity which is in the groove; as line $h$ revolves, the extremity of $(l+c)$ runs up and down the axis of $y$: find the locus of the other extremity.

29. Two rods, whose lengths are $a$ and $b$, $(b > a)$, are connected at one end by a hinge, and the other extremities move in two grooves coinciding with the axes of $x$ and $y$; what is the locus of the hinge when the motions of the rods are so adjusted that $b$ is always inclined to axis of $x$, at an angle double that at which $a$ is inclined to the same axis.

30. A line revolves about the origin, and another about a point $(a, b)$. The point of intersection is always equally distant from the two points of revolution; find its locus.

31. $O, P, Q$, are three points, whereof $O$ is fixed; $Q$ revolves about $O$, and is always at the same distance from it. $P$ revolves about $Q$, so as to be always in the vertical line passing through $Q$, and at a distance from it $= n$ times its distance from the axis of $x$; find the locus of $P$.

32. A sporting mathematician being out with his dog, sees a hare coming towards him in a direction which he takes for that of the axis of $x$; when within a certain distance the dog gives chace. By observing the tract of the dog, which hunts by scent, it is found that the hare describes a curve such that the sum of the co-ordinates of any point is always greater than the distance from the origin by the quantity $a$; find the equation to the curve.

33. On investigating the curve described, the owner of the dog becomes anxious, for he finds that if the hare goes to a distance $a$ from the axis of $y$, and the dog continue to follow, he will never come back. Shew this from your result.

34. A rod (length $l$) revolves about the origin, and a ring moves along the rod in such a manner that its distance from the point of revolution is $n$ times the difference of the co-ordinates of the revolving extremity; find the locus of the ring.

35. Two particles, one of which is fixed, are connected by an extensible thread, which the moving particle stretches in such a manner that, if the length at any time be divided by the constant quantity $a$, the ratio expresses the sine of the thread's inclination to the axis of $x$; find the locus in this and following case.

36. $A$, $B$, $C$, are three particles or points; $A$ is fixed, and $B$ and $C$ move, so that the three are always in a straight line. The distance of $C$ from $A$ is constant, and $B$ moves so that if the square of its distance from $C$ be subtracted from $m$ times the square of the distance of $C$ from $A$, the difference $= n$ times the sum of its co-ordinates multiplied by $m$ times their difference.

37. A particle is observed to move in such a manner about a given point (the origin), that if a line be drawn from it at any time making a constant angle $a$ with the axis of $x$, the difference of the square of this line and the square of the distance from the origin is always equal to the square of the constant quantity $d \cos a$; find the equation to the curve described.

38. Shew from your result that the curve cannot cut the axis of $x$, that it may extend to any distance from that axis (above and below), but cannot approach within a certain distance of it: what is its altitude when in the axis of $y$?

39. Two points, which are always in the same vertical, revolve about another which is fixed; the distance of the upper from the fixed point is always $n$ times the distance between the two revolving points, and the lower point moves in a locus, such that the sum of its co-ordinates is always equal to twice its distance from the upper point; determine the locus.

40. A particle acted upon by several forces, describes a curve such that if at any time a line of constant length $b$ be taken, and the abscissa be increased by $\frac{2}{3}$ of its length, and upon these two a parallelogram be constructed, and there be added to it another parallelogram whose area equals $\frac{4}{3}$ of the square constructed upon the ordinate, then the sum of these two areas is always equal

to a square whose side is the radius vector; find the equation to the curve.

41. A rod revolves about the origin, carrying a string with a weight, which by means of a ring slides down as the rod revolves. The motions are so combined, that the weight is always at the same distance from the axis of $x$. Find the equation to the locus of the ring, the string being of constant length.

42. A line ($l$), greater than the diameter of a circle, revolves about one of its extremities, and from the points in which it intersects the circumference chords are drawn to the other extremity; these are cut by a perpendicular from the revolving extremity of $l$: find the locus of the point of intersection.

43. The axis of $y$ is a groove, in which runs the extremity of a line (length $2b$); $a$ is the length of another line, one of whose extremities is at the origin, and the other is joined to the lower extremity of $2b$; while $a$ revolves, one extremity of $2b$ runs up and down the axis of $y$: find the locus of the middle point of $2b$.

44. A line reaches from $(o, b)$ to $(a, o)$. A point $Q$ runs up this line, and a point $P$, which is always at the same distance ($d$) from $Q$, moves so that its altitude above the axis of $x$ is always twice that of $Q$; find the locus of $P$.

45. When two points move as $P$ and $Q$ in (44), what are the greatest altitudes which $P$ can reach?

46. A comet, whose tail is observed to be $a$ miles in length, approaches the system of the sun and earth, but is found to be always farther from the earth than from a line passing through the centres of earth and sun by once the length of the tail; find the locus of the comet with respect to the earth on the hypothesis that the line passing through its centre remains stationary.

47. A particle moving from the origin is acted on by such forces that at any time its distance from the origin is a geometrical mean between $n$ times the ordinate and $p$ times the abscissa.

48. Explain by a geometrical construction, why to one value of $x$ there are two of $y$. Have these different signs?

49. A particle moves in such a manner that its distance from the origin is always equal to the difference of $m$ times the abscissa and $p$ times the ordinate; find the locus.

50. If $m$ and $p$ in (49) were both proper fractions, the law would manifestly involve an absurdity. Prove that analysis would detect it.

51. Shew also from your result, that it is impossible there can be but one value of $y$ to one of $x$. Have the values of $y$ the same or different signs?

52. $O, P, Q, R,$ are four particles, whereof $O$ and $R$ are fixed, $P$ and $Q$ in motion; $Q$ is always at the same distance from $O$ as $R$ is; and $P$ follows $Q$ in such a manner that the three $Q, P, R,$ are always in a straight line, and the distance of $P$ from $Q$ always $\frac{1}{n}$th part the distance of $Q$ from $R$.

53. The axis of $y$ is a groove of indefinite length, in which runs the extremity of a rod always parallel to the axis of $x$; at a distance $d$ from the origin another rod revolves so as to intersect with the former, the motions being so combined that the portions of the revolving rod between the point of intersection and axis of $x$, is *always* $p$ times the intercepted portion of the other; find the equation to the locus of the point of intersection.

54. On this hypothesis, if $p$ be a proper fraction, shew that the strongest man could not raise the horizontal rod however light above a certain height.

55. Two particles are always in a vertical line, and the distance of the highest from a fixed point given in position, is always a mean proportional between the co-ordinates of the other. Find the locus of the lower point, assuming the fixed point the origin.

56. What would be the equation to the locus, the co-ordinates of the fixed point being $(a, b)$?

57. Having hooked a fish, the rod is held steadily at a certain inclination until the direction of the line (length $s$) is vertical. The rod, continually held in the same position, is then raised gently, and it is observed that the fish approaches the bank by the same distance that the rod is raised. Find the equation to the curve described by the fish, the origin corresponding to the place of the fish when the rod is first raised.

58. Two points are given in position, and a third moves so that its distance from one of them is always $n$ times its distance from the other; find the equation to the curve described by the moving point.

59. A rod (length $2a$) has two weights suspended at its extremities by strings of equal length. The rod is then made to revolve about its middle point. Find the equation to the loci described by the weights.

60. Determine the position of *both* weights at any time from your equation when the strings are unequal, and the point of revolution not the middle point.

61. *Senate-House Problems*, 1843. Find the locus of two tangents to a circle, such that the rectangle under the portions (measured from the centre) cut off by them from a given diameter is always constant.

# CHAPTER IV.

## *The Straight Line.*

1. Def. ***The straight line is the locus of a point moving continually in the same direction.***

In this definition the law of the motion is expressed too generally for the student to derive from it the Equation to the Locus. He might, at first, think that it is only necessary to know the direction in which the point moves; but that this would not be sufficient, a simple case will make evident. Let him conceive several lines drawn parallel to the axis of $x$; these would all have the same direction. Supposing then, for argument's sake, that he knew the equation to the straight line, how is he to distinguish that it refers to one line rather than another, if the only quantity that enters into it (besides the variables $x$, $y$) be the direction which is the same in all?

Some other datum is evidently required; and to determine what this ought to be, he must consider in what respects these and all other parallel lines differ from each other. It is in this: that no two of them pass through the same point; their equations therefore would be distinguishable from each other, if each expressed that the generating point passes through a particular point. Hence *the equation to a line must, besides the direction in which the describing point moves, involve also the co-ordinates of a point through which it or the direction passes.*

Assuming then, as given, the direction of the motion (the angle it makes with the axis of $x$), and the co-ordinates of a point through which that direction passes, the student might proceed as for the loci in the last chapter. But, although he will find that method useful, when easier ones are not obvious, he is always at liberty to adopt any other, however short, if it lead to an equation between ($x$, $y$) and given constants. In the present case *the trigonometrical relation between the sides of a triangle*, viz. that they are proportional to the sines of the opposite angles, *leads at once to an equation* between the variables and constants, *whether the line be referred to oblique or rectangular axes.*

2. To find the equation to a straight line. Let (fig. 19) $OX$, $OY$, be the co-ordinate axes inclined at $i$; $RPQr$ the line whose equation is required; $Q$ the point through which it passes; $(a, b)$ the co-ordinates of $Q$: $(RrX = \alpha)$ the angle $\alpha$ at which the line is inclined to the axis of $x$.

Take $P$ any point in the line; draw $PN$, $QM$ parallel to $OY$, $Qmq$ parallel to $OX$; then

$$PN = y, \quad QM = b, \quad \therefore\ Pm = y - b;$$
$$ON = x, \quad OM = a, \quad \therefore\ Qm = (x - a);$$

also in the triangle $PQm$, since $Pm$, $mq$ are respectively parallel to $OY$, $OX$, the angle $PQm$ equals $\alpha$; angle $QPm = Pmq - PQm = (i - \alpha)$. And, by Trigonometry,

$$\frac{Pm}{Qm} = \frac{\sin PQm}{\sin QPm},$$

or $$\frac{y-b}{x-a} = \frac{\sin\alpha}{\sin(i-\alpha)} \ldots\ldots\ldots\ldots\ldots \text{(A)},$$

which is the equation to a straight line, referred to axes inclined to each other at an angle $i$.

Cor. 1. Let the axes be rectangular, or $i = 90°$, then

$$i - \alpha = 90° - \alpha, \quad \sin(i - \alpha) = \cos\alpha;$$

and (A) becomes $$\frac{y-b}{x-a} = \tan\alpha \ldots\ldots\ldots\ldots\ldots \text{(B)}.$$

The results (A), (B) the student must impress on his memory. The latter suggests a *short rule* for finding the equation to a line referred to rectangular axes, viz. to *express the tangent of its inclination to the axis of $x$, in terms of the co-ordinates of two points through which it passes,* one of them being the given point. From it are easily derived more simple expressions depending upon the position of the point through which the line passes.

For brevity, let $\tan\alpha = t$, or suppose $\alpha = \tan^{-1}t$,* then

$$\frac{y-b}{x-a} = t.$$

Cor. 2. Suppose the line to pass through the origin (fig. 20); then, since the co-ordinates of that point are $(o, o)$ its equation from (B) is

$$\frac{y-o}{x-o} = t, \quad \text{or} \quad \frac{y}{x} = t \ldots\ldots\ldots\ldots \text{(C)}.$$

Cor. 3. Let the line cut the axis of $x$ in a point $d$, (fig. 21); then, since it passes through a point $(d, o)$, its equation from (B) is

$$\frac{y-o}{x-d} = t, \quad \text{or} \quad \frac{y}{x-d} = t \ldots\ldots\ldots \text{(D)}.$$

* This notation the student has already met with in Trigonometry; it signifies that $\alpha$ *is an angle whose tangent is $t$*; so if $\beta = \sin^{-1}s$, $\beta$ is an angle whose sine is $s$, and so on for $\cos^{-1}$, $\sec^{-1}$, &c.

COR. 4. Let it cut the axis of $y$ in a point $e$, (fig. 22); then, since it passes through a point $(o, e)$, its equation from (B) is

$$\frac{y-e}{x-o}=t, \quad \text{or} \quad \frac{y-e}{x}=t \ldots\ldots\ldots\ldots \text{(E)};$$

and the student will easily obtain analogous results, when one or both of the co-ordinates of the given point are negative, bearing in mind to subject the signs of direction to the ordinary rule as observed in page (8). The expression (A), which is the equation to a line referred to oblique axes, will obviously admit of similar transformations.

3. The results (C), (E), (D), (B), transformed out of the fractional form, are respectively

$$\begin{aligned} y &= tx, \\ y &= tx + (e), \\ y &= tx + (-d), \\ y &= tx + (b - ta), \end{aligned}$$

in the last three of which, whatever corresponding values be assigned to $x$ and $y$, the quantities within the brackets always remain of the same magnitude and sign. Hence, if $c$ be assumed as representing a constant quantity of any magnitude or sign, the above results are contained in the two forms

$$\frac{y}{x}=t, \quad \text{or} \quad y=tx \ldots\ldots\ldots\ldots\ldots\ldots (\alpha),$$

$$\frac{y-c}{x}=t, \quad \text{or} \quad y=tx+c \ldots\ldots\ldots\ldots\ldots (\beta);$$

the latter of which, from its generality, is called *the general equation to a line.*

By comparing ($\beta$) with (E), the student will see that the *general equation* is of the form of *the equation to a line, cutting the axis of y.*

The meaning of the letters $t$ and $c$, the letters themselves will recal to his memory;
$c$ is the *co-ordinate* of the point in which it cuts the axis of $y$;
$t$ is the *tangent* of the angle of its inclination to the axis of $x$.

It must be observed, however, that any other letters might be used.

4. Sign of $t$.

Let the line $Tl$ be inclined to the axis of $x$ at an angle $LlX = \pi - a$, and cutting the axis of $y$ in a point $(c)$; its equation from (B) is

$$\frac{y-c}{x} = \tan(\pi - a);$$

but, by Trigonometry, $\tan(\pi - a) = -\tan a$.

Therefore the equation to $Ll$ is $\dfrac{y-c}{x} = -\tan a$,

or $y = -tx + c$.

Hence, in assuming the equation to a line from a given data, the student must be careful to affect the letter which he takes to represent the trigonometrical tangent of its inclination, with a positive or negative sign according to the magnitude of the angle of inclination, following in this respect the rules of Trigonometry.

5. Equation to a line perpendicular to one of the axes.

Let a line $LBl$ (fig. 24) cutting the axis of $y$ in $c$ be perpendicular to that axis, the angle of its inclination to axis of $x = 0$, and its equation from (B) is

$$\frac{y-c}{x} = \tan(0) = o,$$

which shews that no relation can be established between $x$ and $y$. But, in order that a fraction may $= o$, its numerator must $= o$; this condition gives

$$y - c = o, \quad \text{or} \quad y = c \ldots\ldots\ldots\ldots\ldots (\gamma),$$

which equation must be taken for that to $LBl$. Hence, *when a line is perpendicular to either of the co-ordinate axes*, it is represented by an equation in which *the corresponding co-ordinate is equated to a constant quantity*; which will of course be affected with the positive or negative sign according as that axis is cut on the positive or negative side.

6. Every part of a line is properly represented by its equation.

Let $LBol$ (fig. 25) be a line inclined at $(\tan^{-1} t)$, its equation is

$$y = tx + c, \quad \text{or} \quad \frac{y-c}{x} = t,$$

in which $c = OB$.

Now the second side of this equation is positive; it must therefore be shown, since the line lies in the 1st, 2nd, and 3rd quad-

rants, that the first or left-hand side remains positive, whatever quadrant $x$ and $y$ be taken in.

Take three points, $P_1$, $P_2$, $P_3$, and let their co-ordinates be $(P_1N_1, ON_1)$ $(a, b)$; $(P_2N_2, ON_2)$ $(-d, e)$; $(P_3N_3, ON_3)$, $(-f, -g)$, respectively: then, substituting these values of the variable co-ordinates in the equation to the line, the equation to $P_1$ is

$$\frac{a-c}{b}=t;$$

but $a > c$, therefore $a-c$ is positive, therefore the first side is positive. The equation to $P_2$ is

$$\frac{e-c}{-d}=t;$$

but $c > e$, therefore $(e-c)$ is negative, and the numerator and denominator being both negative, the first side remains positive.

The equation to $P_3$ is

$$\frac{-g-c}{-f}=t, \quad \text{or} \quad \frac{-(g+c)}{-(f)}=t,$$

and the left-hand side is positive as before.

7. To trace a line from its equation.

The straight line being the locus of a point moving continually in the same direction, if $P$, $Q$, $R$, (fig. 26) be three points in $Ll$, the direction in which the generating point has moved from $Q$ to $R$, is the same as from $P$ to $Q$. But this direction determines the inclination of the line to the axis of $x$, hence the inclination of the part $QR$ to the axis of $x$ is the same as the inclination of the part $PQ$; in other words, as Euclid states in the definition, *the right (straight) line PQR lies evenly between its extreme points.*

Thus, then, if the inclination to the axis of $x$ of one part of a line be known, that of an adjacent part is known, and therefore that of the whole line is known. Consequently, to trace a line represented by a given equation, it is only necessary to determine the position of two points in it. For the line which lies evenly between these points will be a part of the required line, and producing it indefinitely both ways, the parts of the locus corresponding to any positive or negative values of the variables are traced.

The two points most readily determined are those in which the line cuts the co-ordinate axes; to ascertain which it is only necessary to substitute in the equation to the line, those values

of the co-ordinates which represent a point as situated any where in either of the co-ordinate axes. Thus when a point is anywhere in the axis of $y$, its co-ordinates are $(o, y)$ and $(x, o)$ when in the axis of $x$. Let the equation to the line be

$$y = tx + c,$$

make $x = 0$; then the equation to the point in which it cuts the axis of $y$ is

$$y = c,$$

make $y = o$, and the equation to the point in which it cuts the axis of $x$ is

$$o = tx + c, \quad \text{or} \quad x = -\frac{c}{t},$$

that is to say, the axis of $x$ is cut on the left-hand side.

Hence if $OP$ be taken on the left of $O$ and of a length $= \frac{c}{t}$, and $OQ$ be taken $= c$, the line $PQ$ produced indefinitely towards $L$ and $l$ is the locus of the proposed equation.

8. The locus of the indeterminate equation of the 1st degree between two variables is the straight line.

An equation between variable quantities and constants is said to be of the first degree when no term involves either of the variables to a greater power than the first; and two equations are said to be of the same form when they involve constant and variable quantities in the same manner.

An equation is said to represent a locus when its form is the same as any one of the forms of the equation to that locus, or admits of being put into that form.

The indeterminate equation of the first degree in its most general form is

$$Ax + By + C = o \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

dividing by $B$, and transposing, this becomes

$$y = -\frac{A}{B}x - \frac{c}{B} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

The general equation to a straight line is

$$y = tx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

and both $t$ and $c$ may be negative; on this hypothesis it becomes

$$y = -tx - c,$$

which agrees in form with (2). Hence (2) represents a line. But (2) is derived from (1), therefore a straight line is the locus of (1).

Before proceeding to the problems, the student is advised to turn to the examples at the end of the chapter, and to acquire some dexterity in tracing lines and figures from given equations.

### *Problems on the straight line.*

The general equation to a line being

$$\frac{y-c}{x} = t = \tan\ (\angle \text{ of inclination}),$$

it may be the object of a problem, either to find the tangent of the angle of inclination of a line satisfying certain conditions, or to find the values of $x$ and $y$, corresponding to any particular point in it.

The conditions which a line may satisfy with respect to its inclination are either to be parallel to another line, or to cut it at a given angle, or to pass through certain points. The equation to a line so drawn may be easily found, having given the equation to the second line or the co-ordinates of the points. Prob. 1, 2, 3.

Again, many lines may be drawn differently inclined to the axis of $x$, and yet all passing through the same point in the axis of $y$. See (fig. 35.) If we suppose these lines to intersect with another line $Ll$, the co-ordinates of the point of intersection will be different in each case; but having given the equations to the intersected and intersecting lines, the co-ordinates of the point of intersection may be ascertained.

PROB. 1. The co-ordinates of a point are $(a, b)$, and the equation to a line $\frac{y-c}{x} = t$; find the equation to another line passing through $(a, b)$ and parallel to the given line.

Let $\theta$ be the angle of inclination of the given line; then, since it passes through $(a, b)$, its equation will be

$$\frac{y-a}{x-b} = \tan\theta.$$

But since it is parallel to the given line,

$$\therefore \tan\theta = t;$$

hence the required equation is

$$\frac{y-a}{x-b} = t,$$

$$\text{or}\quad \frac{y-c_1}{x} = t, \quad \text{making } c_1 = b - ta.$$

Hence, when lines are parallel to each other, their equations are distinguished by the co-ordinates of the points through which each passes.

PROB. 2. The co-ordinates of a point are $(a, b)$, and the equation to a given line $\frac{y-c}{x} = t$; find the equation to another line passing through the given point, and making an angle $\theta$ with the given line.

When a line inclined to the axis of $x$ at $\alpha$ is cut by another which makes with it an angle $\theta$, the inclination of this second line to the axis of $x$ equals the sum or difference of the angles $\alpha$ and $\theta$, according as $\theta$ is positive or negative, measuring the angle always the same way. This is obvious from (fig. 27), where supposing $OP$ inclined at $\alpha$ to $OX$, and to bisect the angle $SOs$ equals $2\theta$, then

$$\text{the inclination of } SO = POX + POS = \alpha + \theta,$$
$$\ldots\ldots\ldots\ sO = POX - POs = \alpha - \theta.$$

Let $\alpha$ be the inclination of the given line to the axis of $x$, and suppose $\theta$ a positive angle, then the inclination of the line whose equation is required is $(\alpha + \theta)$; and since it passes through $(a, b)$, its equation is

$$\frac{y-b}{x-a} = \tan(\alpha + \theta);$$

by Trigonometry,

$$\tan(\alpha + \theta) = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta}.$$

If for brevity $\tan\theta$ be represented by $\tau$, then, since $t = \tan\alpha$, the equation becomes

$$\frac{y-b}{x-a} = \frac{t+\tau}{1-t\tau},$$

$$\text{or}\quad \frac{y-c_1}{x} = \frac{t+\tau}{1-t\tau} \text{ assuming } c_1 = b - \frac{t+\tau}{1-t\tau}a.$$

COR. 1. Let $\theta$ be a negative angle; then in the same manner the equation to the second line will be found to be

$$\frac{y-b}{x-a} = \frac{t-\tau}{1+t\tau}.$$

COR. 2. Let $\theta = 90°$, or let the line whose equation is required be perpendicular to the given line; its equation would be

$$\frac{y-b}{x-a} = \tan(\alpha + 90);$$

by Trigonometry, $\tan(a + 90) = -\cot a = -\frac{1}{\tan a}$,
therefore the required equation is

$$\frac{y-b}{x-a} = -\frac{1}{t}.$$

This relation between the equations to two lines perpendicular to each other, viz. that if one be inclined to the axis of $x$ at $\tan^{-1}(t)$, the other is inclined to the same axis at $\tan^{-1}\left(-\frac{1}{t}\right)$, the student must impress on his memory.

Cor. 3. $y = t_1 x + c_1$, $y = t_2 x + c_2$, being the equations to two given lines, if it were required to ascertain the angle at which they intersect, then, since this angle must be the difference of the angles at which they are inclined to the axis of $x$, (see fig. 35,) its value, calling it $\theta$, would be obtained from the trigonometrical equation

$$\tan\theta = \tan(\tan^{-1} t_1 - \tan^{-1} t_2),$$
$$= \frac{t_1 - t_2}{1 + t_1 t_2}.$$

Prob. 3. The co-ordinates of two points are $(a_1, b_1)$, $(a_2, b_2)$; to find the equation to a line passing through these points.

Let $\phi$ be the angle which the line whose equation is required makes with the axis of $x$; then, since it passes through $(a_1, b_1)$, its equation is

$$\frac{y - b_1}{x - a_1} = \tan\phi,$$

$$\text{or} \quad y - b_1 = \tan\phi\,(x - a_1) \;\ldots\ldots\ldots\; (1);$$

also, since it passes through $(a_2, b_2)$,

$$y - b_2 = \tan\phi\,(x - a_2) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence subtracting (1) from (2),

$$b_1 - b_2 = \tan\phi\,(a_1 - a_2),$$

$$\text{or} \quad \tan\phi = \frac{b_1 - b_2}{a_1 - a_2};$$

which determines $\tan\phi$. Substituting in (1) or (2), the required equation is

$$y - b_1 = \frac{b_1 - b_2}{a_1 - a_2}(x - a_1),$$

$$\text{or} \quad y - b_2 = \frac{b_1 - b_2}{a_1 - a_2}(x - a_2).$$

PROB. 4. The equations to two lines which intersect are $y = t_1x + c : y = t_2x + c_2$; to find the co-ordinates of the point of intersection.

**The equation to a line expresses the relation which obtains between the co-ordinates of *every point* in it; and, since each line passes through the point of intersection, that point may be considered as belonging to, or situated in, each line; consequently, by substituting its co-ordinates for the indeterminate quantities $(x, y)$ in the equation to each line, two independent equations will be obtained, which are sufficient to determine their values.**

Let $(X, Y)$ be the co-ordinates of the point of intersection; then, since it is situated in each line, the values of $X$ and $Y$ must be such as to satisfy the equations to both lines. Hence

$$Y = t_1X + c_1 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$Y = t_2X + c_2 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The values of $Y$ obtained from (1) and (2) must be the same; hence, equating $\quad t_1X + c_1 = t_2X + c_2$,

$$\therefore X = \frac{c_2 - c_1}{t_1 - t_2} \ldots\ldots\ldots\ldots\ldots (3).$$

This value of $X$ being substituted in (1) or (2) will give the value of $Y$; thus, from (1),

$$Y = t_1 \frac{c_2 - c_1}{t_1 - t_2} + c_1$$

$$= \frac{t_1c_2 - t_2c_1}{t_1 - t_2};$$

the same value of $Y$ would be obtained by the substitution of (3) in (2).

PROB. 5. The co-ordinates of a point are $(a, b)$, the equation to a line $y = tx + c$; to find the length of the perpendicular drawn from the point on the line.

Let (fig. 28) $Q$ be the point whose co-ordinates are $(a, b)$,
$RPt$ be the line whose equation is $y = tx + c$,
from $Q$ draw $QR$ perpendicular on $RPt$, and $QN$ perpendicular to the axis of $x$,

$$\text{then} \quad QN = b; \quad ON = a \ldots\ldots\ldots\ldots\ldots (1).$$

Now in the triangles $RQP$, $NPt$,

$$\angle tPN = \angle QPR; \ \angle QRP = 90^\circ = \angle PNt:$$

$$\therefore \angle PQR = \angle PtN = \tan^{-1} t \text{ from equation to line;}$$

by Trigonometry,

$$\frac{QR}{QP} = \cos PQR = \frac{1}{\sec PQR} = \frac{1}{\sqrt{(1 + \tan^2 PQR)}}$$

$$= \frac{1}{\sqrt{1 + t^2}} \text{ from equation to line;}$$

$$\therefore QR = \frac{QP}{\sqrt{(1 + t^2)}} \ldots\ldots\ldots\ldots\ldots (2);$$

$$\text{but} \quad QP = QN - PN = b - PN \ldots\ldots\ldots (3),$$

and $P$ being a point $(x, y)$ in the given line, its equation gives

$$y = tx + c$$

$$\text{or} \quad PN = t . ON + c$$

$$= ta + c \text{ from (1);}$$

substituting this in (3) and the result in (2),

$$\therefore QR = \frac{b - ta - c}{\sqrt{(1 + t^2)}}.$$

COR. Had $Q$ been below the line, as in fig. (29), the expression for $QP$ would have been

$$QP = PN - QN$$

$$= ta + c - b$$

$$= -(b - ta - c),$$

$$\text{and therefore} \quad QR = -\frac{b - ta - c}{\sqrt{(1 + t^2)}};$$

which result is comprehended in the former, since the radical sign always involves the double sign $(\pm)$.

The student must remember this expression, and bear in mind that *the denominator* involving $t$ to the square *is not affected in its value when t is negative;* also that the form and value of *the numerator can be readily produced by transposing* the terms in *the equation to the line on the left hand side,* and *substituting,* instead of $x$ and $y$, *the co-ordinates of the given point.* The expression thus obtained is then to be affected with a positive or *negative sign,* according as the point is above or *below the line.* See page (57) for Problems.

*Examples on the Construction or Tracing of Lines.*

The method of constructing a line from the equation $y = tx + c$, which has been explained in (7), must always be resorted to when, instead of $t$ and $c$, which are then taken as the representatives of constant quantities, the values of those quantities are given in express terms.

Now $c$ stands for a line of a certain length, and $t$ for the tangent of an angle; but *the tangent of an angle*, as the student has seen in Trigonometry, *is a ratio*, and is expressed by *a number*, and length is expressed in inches or multiples of an inch. Supposing, then, it were required to construct a line cutting the axis of $y$ in a point distant $-2$ inches from the origin, and inclined at an angle $\tan^{-1}(4)$; the equation to such a line would be

$$y = 4x - 2 \text{ inches} \quad \ldots\ldots\ldots\ldots\ldots \quad (1),$$

and putting $y = o$, the point in which the line cuts the axis of $x$, would be

$$x = \frac{2 \text{ inches}}{4} = \tfrac{1}{2} \text{ inch};$$

and if a line of any length were drawn through the points $(o, -2 \text{ inches})$ $(\frac{1}{2} \text{ inch}, o)$ the co-ordinates of every point in it would strictly satisfy (1), as the student may easily verify by actual construction and measurement. But in a line passing through $(o, -2 \text{ inches})$, and cutting the axis of $x$ in a point differing ever so little from $(\frac{1}{2} \text{ inch}, o)$, there could not be found a point whose co-ordinates would satisfy (1), and therefore such a line could not be the locus of that equation. If, however, the respective values of $t$ and $c$ are not given, actual measurement is out of the case; and all the accuracy which can be arrived at, in constructing from an equation, will consist in making the line to incline to the *right* or *left*, and to cut the axis of $y$ *above* or *below* the origin. These two circumstances, as stated in (4), *depend upon the sign of $t$ and $c$* when the equation to the line is *of either of the forms* (B), $(\alpha)$, $(\beta)$, or $(\gamma)$; therefore, to avoid mistake at first, the student had better put the proposed equation under whichever of these forms it admits of, (the fractional is the best); connecting the constants with the corresponding variable, and *taking care that the variables have the same place as in those forms.* Without attention to this last particular he might be led into error. The equation (B) for example

$$\frac{y - b}{x - a} = t$$

might be written

$$\frac{b - y}{x - a} = -t, \quad \text{or} \quad \frac{y - b}{a - x} = -t;$$

and unless he attended to the left hand side, he would take the two last to represent lines inclined from left to right. This would be wrong, for the equation still represents the same line, it not having been affected by changing the signs of both sides.

($\alpha$) ($\beta$) ($\gamma$) admit of similar observations.

The student bearing in mind what has been said, will find no difficulty in describing any of the figures represented by the sets of equations which follow. Let him *draw a line of indefinite length for each equation,* beginning with such as are of the form ($\gamma$) representing a line perpendicular to either of the axes, and then the parts of each included between the points of intersection will form the figure required.

When lines have different inclinations, this is expressed by $t_1, t_2, t_3 \ldots$ instead of using different letters. When $t$ is the same in the equations to different lines, these lines have the same inclination, and must therefore be drawn parallel to each other in the figure. The relation between $\tan^{-1}(t)$ and $\tan^{-1}\left(-\frac{1}{t}\right)$ is explained (page 48); but as the student is advised to set about these constructions so soon as he has read to Art. 8, he may for the present omit the examples involving $\frac{1}{t}$.

Ex. 1. To construct the figure represented by the following set of equations,

$$1 = \frac{tx}{y}, \quad x - a = 0, \quad y - tx - b = 0, \quad x = 0;$$

putting the equations under known forms, they become respectively

$$\frac{y}{x} = t, \quad x = a, \quad \frac{y - b}{x} = t, \quad x = 0.$$

The first represents a line inclined from right to left, and passing through $(o, o)$ or the origin, by form ($a$). The second, a line perpendicular to the axis of $x$, by form ($\gamma$). The third, a line inclined from right to left, (parallel to the first, because $t$ is the same,) and passing through $(o, b)$, *i.e.* cutting the axis of $y$ in ($b$), by form ($\beta$). The fourth is of the form of a line perpendicular to the axis of $x$, and from the value of $x$ evidently represents the axis of $y$.

Hence, taking $ON$ (fig. 30) to represent $a$, $OM$ to represent $b$, a line through $N$ perpendicular to the axis of $x$ will be the locus

of the second equation. Let this line be *SNR*: the line *POQ* passing through the origin will be the locus of the first equation; and the line *TMV* passing through *M*, and parallel to *POQ*, will be the locus of the third equation.

*A*, *B*, *M*, *O*, are the points of intersection, hence the figure required is *OABM*.

Ex. 2. Trace the path of a point moving so as to describe the following lines,

$$y - tx = 3b, \quad y = \tfrac{7}{2}b, \quad y - 2b = tx, \quad y = 3b, \quad y = tx;$$

put in the known forms, these become

$$\frac{y - 3b}{x} = t, \quad y = \frac{7}{2}b, \quad \frac{y - 2b}{x} = t, \quad y = 3b, \quad \frac{y}{x} = t.$$

The loci of the first, third, and fifth, are parallel lines, the last of which passes through the origin, the others cut the axis of $y$ in $3b$, $2b$, respectively. The loci of the second and fourth are lines perpendicular to the axis of $y$. Hence assume (fig. 31) $OA = b$, and take the points $B$, $C$, $D$, distant from $O$, $2b$, $3b$, $(3 + \frac{1}{2})b$, respectively; then assuming $\angle VOX = \tan^{-1} t$, the lines *PCQ*, *RBS*, *TOV*, being parallel, will be the loci of the first, third, and fifth equations, and the lines *rDa*, *tCd*, which are perpendicular to axis of $y$ will be the loci of the second and fourth. The points of intersection are *abcd*, hence the path of the point is *CabcdO*.

If the *length of the path* were required, the student would have to find, by the formula in the first chapter for the distance between two points, *the length of all its parts* *Ca*, *ab*, *bc*, *cd*, *dO*, which must then be *added together*. But this he cannot do before he has seen how to determine the co-ordinates of the point of intersection of two given lines.

## CONSTRUCTIONS, &c.

The numbers placed to each set of equations, refer to the figures in the plate.

1. Trace the path of a point which, starting from $(a, o)$, returns to that position after describing the lines represented by the following sets of equations:

(44) $y = t_1(x - a),\ \ y = t_2x,\ \ x = a,$

(39) $y = t_2(x - a_1),\ \ x = a_2,\ \ y - t_2a_1 = t_2x,\ \ t_2 > t_1,$

(43) $y + x = a,\ \ y = x + b,\ \ x = a,$

(34) $y = t_1x - t_1a,\ \ t_1y = -(x - a),\ \ y = 0.$

2. Find the perimeters and areas of the figures described by the point moving as in (1), and the co-ordinates of the points in which the different lines intersect.

3. A point starting from $(o, o)$, returns to it after moving in the following lines. Required the figure of each path.

(49) $y = -t_1x,\ \ y = -t_2x + b,\ \ y = b,\ \ y = t_2(x - a),\ \ y = 0,$

(30) $y = tx,\ \ x = a,\ \ y - tx = b,\ \ x = 0,$

(36) $0 = t_1x - y,\ \ y = b_2 - t_2x,\ \ y = b_1,\ \ \left.\begin{array}{r} t_2y - x - t_2b_2 = 0 \\ t_1y + x = 0 \end{array}\right\} b_1 > b_2,$

(50) $y = tx,\ \ ty = -x + b_2t,\ \ y = tx + b_2,\ \ ty = -x.$

4. A point starting from $(o, 2a)$ reaches $(2a, o)$, after describing a path represented by the following equations,

(38) $y + t_1x - 2a = 0,\ \ y = t_2x,\ \ y = t_3x + \dfrac{a}{2},\ \ y = -t_1x + 2a.$

Trace the path, find its length, and determine the area included between it and the co-ordinate axes.

5. A point starting from $(o, o)$ comes to the axis of $y$, after moving as follows,

(46) $y - x = 0,\ \ y = tx + b,\ \ t(y - b) + x = 0,\ \ 2b - y = 0;$
trace the path and find its length.

6. $(p, q)$, $(r, s)$, where $q$ is $>$ than $s$ and $p <$ than $r$, are the co-ordinates of two points referred to rectangular axes; trace the path of a particle* moving as follows

(54) $yp - qx = 0,\ \ (y - b)(r - p) + (q - s)x = 0,\ \ x - r = 0;$
determine the point of intersection of first and last directions.

7. Describe the figures represented by the following equations:

(53) $y = tx + 3b,\ \ y = 3b,\ \ y = tx + 2b,\ \ y = 2b,\ \ y = tx + b,\ \ x = 0,$

(31) $y = tx + 3b,\ \ 2y = 7b,\ \ y = tx + 2b,\ \ y = 3b,\ \ y = tx,$

(48) $x = 0,\ \ y = x + 2b,\ \ x = a,\ \ y = x,\ \ x = 2a,\ \ y = 0,$

* A particle is a physical point.

8. Trace the path, find its length, and the area included between it and the co-ordinate axes when a particle moves as follows,

(41) $y = 3a, \quad y - x = 0, \quad y = tx + a, \quad x = 3a, \quad t < 1.$

9. $(p, q)(r, -s)$ are the co-ordinates of two points referred to rectangular axes; trace the path of a particle, which, starting from the axis of $x$, reaches the origin after moving as follows,

$$y = t_1(x - a), \quad y = -t_2x + c, \quad y = -t_3x;$$

determine $t_1, t_2, t_3$, and $c$, it being given that the portion of the path represented by the second equation is bounded by the given points.

10. Find the area included between the path of the particle and the intercepted portion of the axis of $x$.

11. A particle starting from the origin returns to it after moving in the following directions,

(42) $y = t_1x, \quad y = t_2x + b, \quad x = 0;$

the first and second directions intersect in a point $(-c, -d)$; the second direction also passes through $(-e, f)$: determine $t_1, t_2$, and $b$, and trace the path.

12. A particle moves as follows,

(37) $y + t_1x = 0, \quad x = c, \quad y = -t_2x;$

the second portion of the path is bounded by the points $(c, -d)$ $(c, -2d)$; determine $t_1, t_2$.

13. A particle starts from the origin, and returns to it after moving as follows,

(51) $y = t_1x, \quad 0 = tx - (b + y), \quad 0 = t_1x - b_2 - y, \quad y - t_2x = 0;$

the first direction passes through $(-c, -d)$ $\left(2c - \frac{d}{2}\right)$: determine $t_1, t_2, b_1, b_2$, trace the path, and find the area of the figure described.

14. A particle starting from $(d, o)$ moves as follows,

(47) $y = -t_1x + b, \quad y = -t_2x, \quad t_2y = (x - a);$

trace the path. The point $(-c, d)$ is common to the first and second portions of the path, and the second also passes through $(e, -f)$; determine $t_1, t_2$, and $b$, and find the area of the figure described.

15. A particle starts from $(a, o)$, and returns to it after moving as follows,

(40) $y = t(x - a)$, $y + d = 0$, $x + c = 0$, $y - d = 0$, $ty = td - x$;

trace the figure described, and determine its perimeter; also $t$, it being given that the first direction passes through $(-c, -d)$.

16. Find the area of the figure described by a particle moving as in 15.

17. A particle starts from $(a, -b)$, and returns to that point after moving as follows,

(32) $x = a$, $y = tx$, $ty + x = 0$;

describe the path and find its perimeter: determine $t$, the second direction passing through $(d, e)$.

18. A point starting from $(0, -2a)$ comes to $(0, -\frac{1}{2}a)$ after moving as follows,

(33) $y = x - 2a$, $x = 2a$, $y = x - a$, $y = t(x - a)$;

the second and third directions intersect in a point $(2a, a)$. Trace the path, and find the area included between it and the intercepted portion of the axis of $x$.

19. $t_1$, $t_2$, $t_3$, ... being the tangents of angles $< 90°$, $c_1$, $c_2$, ... points in the axis of $y$, form the equations to the following figures,

47, 57, 46, 41, 40, 44, 43, 34, 50, 42.

20. A telegraph is composed of a fixed vertical rod, the upper extremity of which coincides with the middle point of a moveable rod, and is so fastened to it that the latter may revolve in the vertical direction: at the extremities of this second rod are fastened two others in a manner that will also allow of their revolving in the vertical direction. Supposing the fixed vertical rod to coincide with the axis of $y$, and its length to be $b$, trace the signals indicated by the telegraph when the following equations represent the positions of the moveable rods.

| | Centre. | Right. | Left. |
|---|---|---|---|
| (60) | $y = tx + b$, | $x = -a$, | $y = b_2$. |
| (69) | $x = 0$, | $y = tx + b_2$, | $y = tx + b_1$. |
| (71) | $y = -tx + b$, | $x = a$, | $x = -a$. |

| | | |
|---|---|---|
| (67) $x = 0,$ | $y = b_2,$ | $y = -tx + b_1.$ |
| (65) $y = tx + b,$ | $y = b_3,$ | $y = b_2.$ |
| (75) $y = b,$ | $y = -tx + b_2,$ | $y = \frac{1}{t} x + b_3.$ |
| (62) $y = tx + b,$ | $y = t_2 x,$ | $y = -t_3 x.$ |
| (70) $y = -tx + b,$ | $y = tx + b_2,$ | $y = -t_2 x + b_3.$ |
| (74) $y = -tx + b,$ | $y = t_2 x,$ | $y = -t_3 x.$ |
| (68) $y = b,$ | $x = a,$ | $y = -tx.$ |
| (72) $y = b,$ | $x = a,$ | $y = tx + b_2.$ |
| (66) $y = tx + b,$ | $y = -\frac{1}{t} x + b_2,$ | $y = -t_2 x - b_3.$ |
| (73) $x = 0,$ | $x = 0,$ | $y = b_2.$ |
| (64) $y = b,$ | $y = tx,$ | $y = -\frac{1}{t} x + b_2.$ |
| (63) $y = tx + b,$ | $y = -\frac{1}{t} x + b_2,$ | $y = -\frac{1}{t} x.$ |

*Problems on the Straight Line.*

To find the area of the figure alluded to in (8).

On constructing the lines, the student will find that the point moves so as to describe *ABCDE*, (fig. 41), the direction of *BC* passing through the origin.

In this and similar cases, the first thing is to divide the figure into triangles, taking care that they be as few as possible, and that one side of each triangle be either coincident with or parallel to one of the co-ordinate axes, the angular points being the points of intersection of the lines. It will then only be necessary to determine the co-ordinates of the points of intersection, from which the length of the perpendicular and base of each triangle may be determined without employing ($\delta$), Chap. I. When, however, the lines are such as not to admit of this convenient division, recourse must be had to that formula.

Producing *BC* to the origin, and drawing a line from *C* to *E*, the figure *ABCDEO* is divided into three triangles, and

$$\text{area required} = \triangle ABO + \triangle OCE + \triangle DCE.$$

Let $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, be the co-ordinates of *B, C, D,* respectively, then

$$\text{the triangle } ABO = \tfrac{1}{2}(AO \times AB) = \tfrac{1}{2}(x_1 \times y_1);$$

so the triangle $OEC = \frac{1}{2}(x_3 \times y_2)$,

the triangle $DCE = \frac{1}{2}\{y_3 \times (x_3 - x_2)\}$;

$\therefore$ area required $= \frac{1}{2}\{x_1 y_1 + x_2 y_2 + y_3 (x_3 - x_2)\}$.

The values of $x_1$, $y_1$, &c. must now be found as in Prob. (5), and substituted in the above.

Prob. 1, p. (62), fig. 55.

Let $AOB$ be the equilateral triangle; then $OB = a$; $OP$ the line inclined at 45° intersecting the side $AB$ in $P$, and let $(x_1, y_1)$ be the co-ordinates of $P$; because the triangle is equilateral,

$$\therefore \angle AOB = \angle ABO = \angle OAB = 60°;$$

and $AB$, measuring the angle of inclination as usual, is inclined to the axis of $x$ at $(\pi - 60°)$, its equation is therefore

$$\frac{y}{x-a} = \tan(\pi - 60°) = -\tan 60°,$$

$$\therefore y = -\tan 60° (x - a) \ldots\ldots (1);$$

and $\because \tan 45° = 1$,

$$\therefore \text{the equation to } OP \text{ is } y = x \ldots\ldots\ldots\ldots (2);$$

these lines both pass through $(x_1, y_1)$, therefore the values of these quantities must satisfy (1), (2). Hence

$$y_1 = -\tan 60° (x_1 - a),$$

$$y_1 = x_1;$$

whence $x_1 (1 + \tan 60°) = a \tan 60°$,

$$\therefore x_1 = a \cdot \frac{\tan 60°}{1 + \tan 60°};$$

by Trigonometry, $\tan 60° = \sqrt{3}$,

$$\therefore x_1 = a \cdot \frac{\sqrt{3}}{1 + \sqrt{3}} = y_1, \text{ from (2).}$$

Prob. 26, p. (65), fig. 56.

Let $AOBC$ be the square, $AO = a$, $AP = \frac{a}{p}$; then $CP$ is the line whose equation is required.

This equation will be of the form

$$y = tx + c \ldots\ldots\ldots (1);$$

but since the line passes through $C(-a, o)$, and through $P\left(-\frac{a}{p}, a\right)$,

$$\therefore\ o = -ta + c \ldots\ldots\ldots (2),$$

$$a = -t\frac{a}{p} + c \ldots\ldots\ldots (3).$$

Substituting (2) in (3), $a = -t\frac{a}{p} + ta$,

$$\text{whence}\quad t = \frac{p}{p-1} \ldots\ldots\ldots (4);$$

and substituting (4) and (2) in (1), the required equation is

$$y = \frac{p}{p-1}(x + a).$$

Prob. 36, p. (66), figs. 57, 58.

Let $OPQ$ be the line inclined at 45°,

$RPS$ .................. $\beta$ to $OPQ$,

$\theta$ .... angle at which $RPS$ is inclined to the axis of $x$,

$c$ .... point in which it cuts the axis of $y$;

then $\theta = (45° + \beta)$,

$$\therefore\ \tan\theta = \tan(45° + \beta) = \frac{1 + \tan\beta}{1 - \tan\beta} \ldots\ldots (A),$$

and the equation to $RPS$ is

$$y = \frac{1 + \tan\beta}{1 - \tan\beta}x + c.$$

Let $x_1, y_1$, be the co-ordinates of $P$; then, since $RPS$ passes through $P$, its equation is also

$$y - y_1 = \frac{1 + \tan\beta}{1 - \tan\beta}(x - x_1);$$

but $OP = d$ by question, and is inclined at 45°,

$$\therefore\ y_1 = x_1 = \frac{d}{\sqrt{2}},$$

$$\therefore\ y - \frac{d}{\sqrt{2}} = \frac{1 + \tan\beta}{1 - \tan\beta}\left(x - \frac{d}{\sqrt{2}}\right) \ldots\ldots (B);$$

and to find where it cuts the axis of $y$, putting $x = 0$,

$$y = \frac{d}{\sqrt{2}}\left(1 - \frac{1 + \tan\beta}{1 - \tan\beta}\right) \ldots\ldots\ldots (C).$$

The student has now to consider the effect of the different values of $\beta$ upon the expressions (A), (B), (C).

Let $\beta = 45°$, then $\theta = (45° + 45°) = 90°$,

$$\therefore \tan\theta = \text{infinity} = \frac{1}{0}.$$

Substituting this in (B),

$$y - \frac{d}{\sqrt{2}} = \frac{1}{0}\left(x - \frac{d}{\sqrt{2}}\right),$$

$$\therefore \frac{y - \frac{d}{\sqrt{2}}}{\frac{1}{0}} = \left(x - \frac{d}{\sqrt{2}}\right),$$

a quantity divided by infinity = 0,

$$\therefore 0 = x - \frac{d}{\sqrt{2}}, \quad \text{or} \quad x = \frac{d}{\sqrt{2}},$$

which is the equation to a line perpendicular to the axis of $x$, as the line $RPS$ then is.

The student knows that two parallel lines never intersect; therefore that when $RPS$ is perpendicular to the axis of $x$, it cannot cut the axis of $y$. This result is indicated by (C), for that equation gives

$$y = \frac{t}{\sqrt{2}}\left(1 - \frac{1}{0}\right), \quad \text{or} \quad y = \frac{t}{\sqrt{2}}\left(1 + \frac{1}{0}\right);$$

that is, the point of intersection is infinitely distant in both directions.

Let $\beta$ be $< 45°$, then $\tan\beta < 1$, and the denominator in (A) is positive, and therefore the line is inclined from right to left, as in fig. 57.

From the figure it appears that the axis of $y$ is cut on the negative side. This result is also obtained from equation (C); for since $\tan\beta < 1$, therefore $\frac{1 + \tan\beta}{1 - \tan\beta}$, which remains positive, is an improper fraction; and, subtracting it from 1, the result is negative.

When $\beta > 45°$, $\tan\beta > 1$, and the denominator in (A) is negative; therefore $\tan\theta$ is a negative quantity, or the line is inclined from left to right, as in fig. (58); and it cuts the axis

of $y$ on the positive side. This latter result is also obtained from (C), where the quantity within the bracket must remain positive, since $\frac{1+\tan\beta}{1-\tan\beta}$ is negative, and being affected with the negative sign becomes a positive quantity.

Prob. 39, page (67), fig. 59.

Let $OX$, $OY$, be the co-ordinate axes,

$m, n,$ .... points $(-a, b)$ $(-d, -e)$,

$Ll,$ .... line passing through these points,

$P,$ .... point, the perpendicular from which divides the triangle as required.

The equation to $Ll$ will be of the form

$$y = tx + c \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1);$$

and since it passes through $(-a, b)$, $(-d, -e)$,

$$\therefore \quad t = \frac{b+e}{-a+d} = \frac{b+e}{d-a};$$

also, as the co-ordinates of either of these points will satisfy (1),

$$\therefore \quad b = -ta + c, \quad \text{or } c = b + ta.$$

Hence, substituting these values of $t$ and $c$, the equation to $Ll$ becomes

$$y = \frac{b+e}{d-a}x + b + \frac{b+e}{d-a}a;$$

putting successively $x$ and $y = 0$, the points $R$ and $S$ are determined, and $OR$, $OS$ being known, the area of the triangle $ROS$ is known: for brevity, let

$$\text{area } ROS = \tfrac{1}{2}A.$$

Let $(o, y_1)$ be the co-ordinates of the points from which the perpendicular is drawn; then if $p$ be its length,

$$p = \pm\frac{y_1 - c}{\sqrt{(1+t^2)}},$$

and $$\text{area } PTR = \tfrac{1}{2}p\,.\,TR.$$

But from similar triangles $PTR$, $ROS$,

$$\angle\, TPR = \angle\, RSO;$$

$$\therefore \frac{TR}{p} = \tan TPR = \tan RSO = t,$$

$$\therefore\ TR = pt;$$

$$\therefore\ \text{area } PTR = \tfrac{1}{2}tp^2 = \tfrac{1}{2}t\,\frac{(y_1 - c)^2}{1 + t^2}.$$

By question, area $PTR = \dfrac{1}{n}$ area $ROS$,

$$\therefore\ t \,.\, \frac{(y_1 - c)^2}{1 + t^2} = \frac{A}{n},$$

$$\therefore\ y_1 = c \pm \sqrt{\left(\frac{1 + t^2}{t} \cdot \frac{A}{n}\right)};$$

this double value shews that there are two points from which a perpendicular might be drawn to $Ll$ so as to form a triangle, the area of which $= \dfrac{A}{n}$, one above and the other below the point $R$.

From geometrical considerations, the student knows that the perpendicular must be of the same length in both cases: this follows also from the analytical expressions; for if $y_1$ be taken with the positive sign, it is $> c$, and corresponds to the point $Q$, which being above the line, the expression for the perpendicular must be taken with the positive sign. Hence

$$p = \frac{1}{\sqrt{(1 + t^2)}} \sqrt{\left(\frac{1 + t^2}{t}\right) \cdot \frac{A}{n}}$$

$$= \sqrt{\left(\frac{A}{tn}\right)};$$

if $y_1$ be taken with the negative sign, it will correspond to the point $P$; and the expression for $p$ being affected with the negative sign, will give the same value as before.

## PROBLEMS, &c.

1. An equilateral triangle (side $a$) has one of its angular points coinciding with the origin, and a side with the positive axis of $x$; find the co-ordinates of the point in which a line inclined at 45° and passing through the origin cuts the inclined side.

2. The line drawn as in (1) divides the triangle into two parts; what ratio have the areas of these parts?

3. Draw a line from $(-d, o)$ to $(o, -d)$, and from its middle point draw two other lines passing through $(d, o)$, $(o, d)$; find the angle included between these lines and the points in which they intersect the co-ordinate axes.

4. The base of an isosceles right-angled triangle coincides with the axis of $y$, and from $(-a, b)$ perpendiculars are drawn to the sides (produced if necessary); what are their lengths, and where do they intersect the axis of $y$?

5. Given the equations to two lines,

$$y = t_1 x + c, \quad y = t_2 (x - b);$$

find the area of the figure contained between them and the intercepted portions of the axes.

6. An equilateral triangle is placed with one of its sides parallel to the axis of $x$, and the co-ordinates of the angular point nearest the origin are $(b, c)$; find the equations to the three sides, the length of each being $a$.

7. $a, b, c$, are the sides of a triangle whose area $= A$; $p_1, p_2, p_3$, the perpendiculars upon them from $(d, c)$ which divides the triangle into three equal parts. What are the equations to the sides, one angular point being at the origin, and the opposite side cutting axis of $y$ in $g$?

8. Determine the angles of the triangle in (7).

9. A line length $l$ is drawn from the positive axis of $y$ to the positive axis of $x$, and inclined at $a$ to the latter: it is bisected by a line which cuts the intercepted portion of the axis of $x$ in a point one-third of its length from the origin. What is the equation to this line?

10. Three lines cut the axis of $y$ in $a, b, c$; $a$ being $> b$, and $b > c$; their inclinations to axis of $x$ being $\tan^{-1} t_1$, $\tan^{-1} t_2$, $\tan^{-1} t_3$, respectively, where $t_3 > t_2$, and $t_2 > t_1$. Find the area of the triangle, the angular points of which are the points of intersection of the lines.

11. Two equal squares have their diagonals coinciding with the axis of $y$, and a line inclined at 45° to that axis; required the co-ordinates of the points in which the sides intersect.

12. One of the angular points of a right-angled triangle is taken as the origin; the base coincides with the positive axis of $x$, and the hypothenuse is above that axis, $b$ and $p$ being the lengths of the base and perpendicular sides respectively; the equation to a line being $y = \frac{\sqrt{(1-c^2)}}{c} x$, where does a parallel line passing through the other extremity of the hypothenuse cut the base?

13. A line through the origin inclined at $\sin^{-1} s$ intersects another passing through a point $(d, o)$ and inclined to the axis of $x$ at $\cos^{-1} c$; find the co-ordinates of the point of intersection.

14. A square (side $a$) has one side coinciding with the positive axis of $y$, and another with the negative axis of $x$; a line is drawn from a point in the side coinciding with the axis of $x$, distant one $q^{\text{th}}$ of its length from the axis of $y$, to cut the parallel side at one $p^{\text{th}}$ of its length ($p > q$) from the same axis; required the equation to the line passing through these points.

15. How does the line drawn as in (14) divide the square, and what relation must there be between $p$ and $q$ that the parts may be as $1:2$?

16. $y = t_1 x$, $y = t_2 x$, $y = t_3 x$, are the equations to three lines; $p_1, p_2, p_3$, perpendiculars on each of them from $(o, b)$; determine $t_1, t_2, t_3$, so that $p_1, p_2, p_3$, and $b$, may be respectively as 1, 2, 3, 4. Explain the double sign.

17. Two lines which touch a circle intersect and pass through the points $(o, a)$, $(o, b)$ respectively; $r$ is the radius of the circle, and $(c, d)$ the position of the centre; determine at what angle the lines intersect the radii to points of contact cut the axis of $x$ in $e_1$, $e_2$.

18. The diagonal of a square (side $s$) coincides with the axis of $y$; what are the equations to the four sides?

19. The equations to three lines are $x = a$, $x = b$, $y = tx$; find the area included between them and the intercepted portions of the axis.

20. An equilateral triangle being placed with one side coinciding with the axis of $x$ and the angular point at the origin, a line not parallel to the axis of $x$ is drawn cutting the inclined

sides (each = $a$) in parts which have to each other the ratio 3 : 1; find the equation to this line, and determine where it cuts the axis of $x$.

21. A line inclined at 45° coincides with the diameter of a circle, and if from a point $(a, o)$ another line be drawn to the first, it will just touch the circle at the extremity of the diameter; what is the radius of the circle, and what are the co-ordinates of the centre?

22. An isosceles triangle (base $b$), and vertical $\angle$ 30° rests on the axis of $x$, $(2b, o)$ being the farthest angular point; required the equation to a line passing through the origin and bisecting the farther side.

23. A line (length $l$) is drawn from the negative axis of $x$ to the negative axis of $y$, and $p$ is the length of a perpendicular on it from the origin; find the equation to a line passing through the origin and inclined at $\beta$ to the perpendicular.

24. Find the same when the perpendicular has its greatest value, and explain the effect of the double sign when $p$ is not the greatest value of the perpendicular.

25. $s$ is the side of an equilateral triangle, which is described within a circle, with one side parallel to the axis of $x$. This being produced cuts the axis of $y$ in $a$; the other sides being produced pass through $(o, b)$ $(o, c)$; required the radius and co-ordinates of the centre of the circle.

26. A square constructed upon a line $a$, has one side coinciding with the positive axis of $y$, and another with the negative axis of $x$. From the angular point on the axis of $x$ a line is drawn cutting the upper side in a point $\left(\frac{1}{p}\right)^{\text{th}}$ its length from the axis of $y$; required the equation to this line, and the point in which it cuts the axis of $y$.

27. A line (length $l$) being drawn from the positive axis of $y$ to the positive axis of $x$, and inclined at $a$ to the latter, a point is taken in it at a distance one $m^{\text{th}}$ of $l$ from the axis of $y$; through this point a line is drawn cutting the intercepted portion of the axis of $y$ in one $n^{\text{th}}$ of its length from the origin; find the

equation to this line determining $(o, c)$, and explain the effect of the negative sign in the denominator.

28. From $(o, -d)$ a line is drawn to $(c, o)$; find the equation to a parallel line passing through $(d, o)$; where does it cut the axis of $y$?

29. Two lines cut the axis of $y$, one in a point $p$, and is inclined at 45°, the other in a point $q$, and is inclined at $\text{vers}^{-1}(r)$; find the area of the triangle formed by these lines, and the intercepted portion of the axis of $y$.

30. The point $(o, b)$ is the centre of a circle, the circumference of which passes through the origin, and the equation to a chord is $y = tx$; what is the radius of the circle?

31. A part of the axis of $y$ (length $b$) is taken as the base, and on it a right-angled triangle is described each of the sides $= s$; from a point in the axis of $y$ a line is drawn to the right angle, thus dividing the triangle into two parts; determine the point so that the area of one of them may be twice that of the other.

32. $(a, b)\,(2a, b)\,(3a, 2b)$ are the angular points of a triangle; determine a point, the lines from which divide the triangle into three equal parts.

33. Two lines include an angle $\beta$, and one of them is inclined at 45° to the axis of $x$; find the equation to the other line, the direction of both passing through the origin.

34. In travelling along a road two objects are observed in directions making the angles $\alpha$, $\beta$ with the road, and $\alpha > \beta$; at a distance $d$ there is another road at right angles to the first; after proceeding along which a distance $e$, both objects are seen in a direction making an angle $\gamma$ with that road; find the distance of the objects from each other.

35. A triangle, whose area equals $A$, has one side parallel to the axis of $x$, and $(a, b)$ $(c, d)$ are the two farthest angular points; determine the third angular point, and also the points in which the co-ordinate axes are intersected by the locus of the vertices of all triangles, (area $= A$) having for base the side opposite the third angular point.

36. A line passes through the origin, and inclined at 45°, intersects another not passing through the origin; they include an angle $\beta$, and the distance of the point of intersection from the origin is $d$; find the equation to the line not passing through the origin, when $\beta$ is greater, less, or equal to 45°, determining $(o, c)$ in each case.

37. A line through the origin inclined to the axis of $x$ at $\cos^{-1}(c^{-1})$, intersects another whose equation is $y = t(x - d)$; find the area of the triangle formed by these two lines, and the intercepted portion of the axis.

38. A line reaches from $(o, b)$ to $(a, o)$, and a perpendicular is drawn to it from the origin; what is its length?

39. A line, the direction of which passes through $(-a, b)$ $(-d, -c)$, $d$ being greater than $a$, intersects the axes so as to form a triangle; from a point in the positive axis of $y$ a perpendicular is drawn to this line, thus forming another triangle, the area of which is one $n^{\text{th}}$ that of the first; required the length of this perpendicular, and the co-ordinates of the point from which it is drawn. Explain the double sign, and shew that it does not affect the length of the perpendicular.

40. $y = t_1 x$, $y = t_2 x$, are the equations to the lines $SO$, $sO$ (fig. 27); find the equation to $PO$, which bisects the angle $SOs$. Shew from the equation for determining the inclination of $OP$ that there are two lines at right angles to each other which satisfy the question.

# CHAPTER IV.

## *On the Circle.*

DEF. The circle is the locus of a point moving in a plane in such a manner as to be always at the same distance from a fixed point called its centre.

The student knows that this invariable distance determines the magnitude of a radius of the circle; and he cannot but perceive that the readiest way to find the equation to the locus, is to express that the distance between the centre and any point in the locus is equal to this magnitude (which is supposed to be given); that is, *to adapt formula* $\delta$, in the first chapter, by *substituting the length of the radius* for $d$, and *the co-ordinates of the centre* (which must be supposed known) *for those of* $Q$.

1. To find the equation to the circle.

Let $OX$, $OY$ (fig. 77) be the co-ordinate axes, $P$ any point in the circle, $C$ its centre, $(a, b)$ the co-ordinates of $C$. Draw $PN$, $CM$ parallel to $OY$, $Cm$ parallel to $OX$, then $CmNM$ is a parallelogram, and $Cm = MN$, $mN = CM$; also $CM = b$, $OM = a$.

Let $PN = y$, $ON = x$, then $Cm = ON - OM = x - a$,
$$Pm = PN - mN = y - b.$$

Let $r$ be the radius of the circle, then from right-angled triangle $PmC$,
$$(PC)^2 = (Cm)^2 + (Pm)^2$$
$$r^2 = (x-a)^2 + (y-b)^2 \ldots\ldots\ldots\ldots (C_1).$$

COR. 1. Suppose the centre to be at the origin (fig. 78), its co-ordinates then are $(o, o)$, and the equation becomes
$$(x-o)^2 + (y-o)^2 = r^2,$$
$$\text{or} \quad x^2 + y^2 = r^2.$$

COR. 2. Let the centre be a point in the axis of $x$ (fig. 79), its co-ordinates $(a, o)$, then the general equation becomes
$$r^2 = (x-a)^2 + y^2;$$
so, if the centre were at $(o, b)$, the equation would be
$$r^2 = x^2 + (y-b)^2.$$

COR. 3. Let the centre be a point in the axis of $x$, distant ($r$) from the origin, *i.e.* suppose the origin of the co-ordinates to be at the extremity of a diameter coinciding with the axis of $x$ (fig. 80), then its co-ordinates are $(r, o)$, and the general equation becomes
$$r^2 = (x - r)^2 + y^2,$$
which, being expanded and reduced, gives
$$y^2 = 2rx - x^2.$$

COR. 4. And if the origin were at the extremity of a vertical diameter (fig. 81), in which case its co-ordinates would be $(o, r)$, the general equation would give, by a similar process,
$$x^2 = 2ry - y^2.$$

COR. 5. The signs in all these equations will be affected if one or both of the co-ordinates of the centre be affected with the negative sign; for instance, let $(\alpha, -\beta)$ be the co-ordinates of the centre, then, substituting in the general equation, it becomes
$$(x - \alpha)^2 + \{y - (-\beta)\}^2 = r^2,$$
$$\text{or} \quad (x - \alpha)^2 + (y + \beta)^2 = r^2;$$
and similarly in other cases.

2. Having given the equation $x^2 + y^2 + Ax + By + C = 0$, to shew that it may represent a circle.

In addition to what was said before, that to represent a locus an equation must admit of being put into the form of the equation to that locus, it must be observed that the nature or properties of the locus might restrict some or all of the constants in its equation to particular values, and therefore it would be requisite that the constants involved in the proposed equation should have such a relation as to admit of this particular value. This would give rise to an *equation of condition*, without satisfying which, the proposed equation, though susceptible of the required form, could not represent the locus.

Completing the squares and transposing, the equation becomes
$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2}{4} + \frac{B^2}{4} - C \ldots\ldots\ldots (1).$$

The general equation to the circle referred to the rectangular axes is
$$(x - a)^2 + (y - b)^2 = r^2 \ldots\ldots\ldots\ldots (2),$$
and if the co-ordinates of the centre be negative,
$$(x + a)^2 + (y + b)^2 = r^2 \ldots\ldots\ldots\ldots (3).$$

This agrees with (1), therefore (1) represents a circle in which

$$r^2 = \frac{A^2}{4} + \frac{B^2}{4} - C,$$

$r$ being the radius.

Now $r^2$ being a square cannot be a negative quantity; in order therefore, that (1) may represent a circle, it is necessary that

$$\frac{A^2}{4} + \frac{B^2}{4} - C \text{ be positive;}$$

when this condition is satisfied (1), and therefore the proposed equation from which it is derived represents a circle, the radius of which

$$= \sqrt{\left(\frac{A^2}{4} + \frac{B^2}{4} - C\right)},$$

and the co-ordinates of whose centre are

$$\left(-\frac{A}{2}, -\frac{B}{2}\right).$$

Also, from the nature of the case, the radius of a circle must be of some magnitude; hence, whatever suppositions be made concerning the values of $a$, $b$, and $r$, in the equation to the circle, $r$ can never be assumed $= 0$; and therefore, in the proposed equation,

$$\frac{A^2}{4} + \frac{B^2}{4} - C \text{ cannot } = 0.$$

COR. If $C$ in the proposed equation *be negative*, the equation always represents a circle.

The equation $x^2 + y^2 + Ax + By + C = 0$, is an equation of the second degree between two variables, with the following peculiarities:

It does not contain the rectangle $xy$ . . . . . . . . . . . . . . . . . .(1).
Both the squares are affected with the positive sign. . . . . .(2).
. . . . . . . . . . . . . . . . .have 1 for coefficient . . . . . . . . . . . . . . (3).

Hence, when the student meets with an equation of the second degree between two variables, and wishes to know if it represents a circle referred to rectangular axes, he must transpose all the terms on one side, to see if (1)(2)(3) be satisfied; and lastly, look for the relation among the constants.

Ex. Given the equation $y(4y - 16) = 4\{6 - (2x + x^2)\}$.
Transposing, $4y^2 - 16y + 4x^2 + 8x - 24 = 0$,
dividing by 4, $y^2 - 4y + x^2 + 2x6 = 0$;
which satisfies (1)(2)(3).

Also, comparing like terms,

$$\frac{A}{2} = 1\ \frac{B}{2} = -2,$$

$C = -6$, and therefore

$$\frac{A^2}{4} + \frac{B^2}{4} - C = \frac{1+4}{4} + 6,$$

and is positive. Hence the above equation represents a circle whose radius $= \frac{\sqrt{(29)}}{2}$, and the co-ordinates of whose centre are $(-1, 2)$.

The equation $y^2\left(1 + \frac{3}{y}\right) = x - (x^2 + 2)4$,
when transposed, is $y^2 + 3y + 4x^2 + 8 - x = 0$,
which cannot represent a circle, since both squares have not 1 for coefficient.

Again, the equation $2y^2 = x + y + 8\left\{\left(\frac{x}{2}\right)^2 - 1\right\}$,
transposed, becomes $2y^2 - y - x - 2x^2 + 8 = 0$,
dividing by 2, $y^2 - \frac{y}{2} - \frac{x}{2} - x^2 + 4 = 0$,
which cannot represent a circle, since the signs of the squares are different.

3. To trace the circle represented by a proposed equation:

$$y(y - 2) = (11 + x)(4 - x).$$

Before this can be done, two things must be known, viz. the position of the centre, and the length of the radius; and the student ascertains these by inspection when the proposed equation is put into the required form; hence, as a *general rule.*

Write down the general equation to the circle, put the proposed equation into that form, and *equate a, b, and r, to the quantities similarly placed and connected.*

The general equation to the circle is

$$(y - b)^2 + (x - a)^2 = r^2.$$

Transposing and completing squares, the proposed equation becomes
$$y^2 - 2y + 1 + x^2 - 4x + 4 = 11 + 1 + 4,$$
$$\text{or} \quad (y-1)^2 + (x-2)^2 = 16.$$
Hence, equating like quantities,
$$b = 1, \quad x = 2, \quad r^2 = 16, \quad \therefore r = 4,$$
or the co-ordinates of the required centre are (2, 1), and the length of the radius = 4. To describe it: Take (fig. 82) $ON = 2$, $NC = 1$, and $CP = 4$, and, with $CP$ as the radius, describe a circle; this will be the locus of the proposed equation.

Ex. Let the proposed equation be
$$y^2 + 8y + x^2 - 12x + 4 > = 0.$$
Completing squares,
$$y^2 + 8y + 16 + x^2 - 12x + 36 = -47 + 36 + 16,$$
$$\text{or} \quad (y+4)^2 + (x-6)^2 = 5,$$
$$\text{that is} \quad \{y - (-4)\}^2 + (x-6)^2 = 5;$$
$$\text{equating to} \quad (y-b)^2 + (x-a)^2 = r^2,$$
$$a = 6, \quad b = -4, \quad r = \sqrt{5}.$$
Hence, $OX$ and $OY$ being the positive axes, taking $OA = 6$, $AC$ in the negative direction = 4, and $CP = \sqrt{5}$; the circle described, with $CP$ as the radius, is the required locus.

From the figure it appears that the circle does not cut either of the axes, and that it cannot do so appears from the equation when put in the required form, since $\sqrt{5} < 6$ or 4, or the radius is less than either of the co-ordinates of the centre, comparing $a, b, r$, with respect to magnitude only. Had it been asked where the locus of $y^2 + 8y + x^2 - 12x + 47 = 0$ cuts the co-ordinate axes, the student, by the regular process of putting $x$ and $y$ severally $= 0$, would have found
$$y = -4 \pm \sqrt{(-31)},$$
$$x = 6 \pm \sqrt{(-11)},$$
both involving what is called an *impossible* quantity. Hence, clearly from the present case, the meaning to be attached to this expression is, that points determined by such values of their co-ordinates do not exist, and therefore the locus of the above equation does not cut the axes.

When the student wishes to find a line to represent a number such as $\sqrt{5}$, let him decompose it into the sum of two others, whereof one is a square, then make use of the right-angled triangle. Thus, to represent $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$,

$$\text{since } \sqrt{5} = \sqrt{(4+1)} = \sqrt{(2^2+1)},$$

take (fig. 84) $OA = 2$, $AB = 1$, and the perpendicular $OA$; join $OB$ and from the triangle $BOA$,

$$OB^2 = OA^2 + BA^2 = 4 + 1 = 5,$$
$$\therefore OB = \sqrt{5};$$

and similarly if $BC$, $CD$ be taken perpendicular to $OB$, $OC$, respectively, and $= 1$, then

$$OC = \sqrt{(OB^2 + BC^2)} = \sqrt{(5+1)} = \sqrt{6},$$
$$OD = \sqrt{(OC^2 + DC^2)} = \sqrt{(6+1)} = \sqrt{7}.$$

If the required number be decomposed into the difference of two others, the semi-circle must be employed, the angle in which, as the student knows, is a right angle. Thus, required $\sqrt{15}$,

$$\because \sqrt{15} = \sqrt{(16-1)} = \sqrt{(4^2-1)}.$$

Take (fig. 85) $AB = 4$, and describe the semi-circle $BDA$, take $BC = 1$, and describe the arc $CD$. Draw $AD$, $DB$, then, from the triangle $ADB$,

$$AD^2 = AB^2 - BD^2 = 4^2 - 1 = 16 - 1 = 15,$$
$$\therefore AD = \sqrt{15}.$$

4. To find the equation to the circle when referred to oblique axes.

This is merely to adapt ($\delta_2$) in the first chapter.

Let $OX$, $OY$, (fig. 86) be the co-ordinate axes, $i = YOX$ the angle of inclination, $C$ the centre of the circle, $(a, b)$ the co-ordinates of $C$, $P$ any point in the circumference, $(x, y)$ the co-ordinates of $P$, $r$ the radius of the circle.

Draw $CM$, $PN$ parallel to the axis of $y$, $Cmn$ parallel to the axis of $x$, then $CMNm$ is a parallelogram, and $Cm = MN$, also $Pmn = YOX = i$.

$$\therefore Pm = (y - b), \quad Cm = x - a,$$

and, from the triangle $CPm$,

$$CP^2 = Pm^2 + Cm^2 + 2Cm \,.\, Pm \cos(Pmn),$$

$$\text{or} \quad r^2 = (y-b)^2 + (x-a)^2 + 2(y-b)(x-a)\cos i \ldots\ldots (C_2).$$

5. To ascertain the conditions to be satisfied in order that a circle may be represented by the general equation of the second degree between two variables

$$\alpha x^2 + \beta y^2 + \gamma xy + \delta x + \varepsilon y + \zeta = 0 \ldots\ldots\ldots (G);$$

in which ($\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, $\zeta$,) represent constant quantities, positive or negative.

The process here obviously is, to expand the terms of the equation ($C_2$); arrange them according to the powers of the variables, and compare each with the corresponding terms in ($G$).

The equation to the circle referred to the oblique axes is

$$(y - b)^2 + (x - a)^2 + 2(x - a)(y - b)\cos i = r^2.$$

Expanding the terms, and arranging according to the powers of $x$ and $y$, it becomes

$$x^2 + y^2 + 2xy\cos i - 2(a + b\cos i)x - 2(b + a\cos i)y$$
$$+ a^2 + b^2 + 2ab\cos i - r^2 = 0.$$

The coefficients of $x$ and $y$ involve $a$ and $b$, which may be of any magnitude and sign; $\delta$ and $\epsilon$ may, therefore, be of any magnitude and sign. For the same reason the magnitude or sign of $\zeta$ are not restricted. But the three first terms give the following equations,

$$\alpha = 1, \quad \beta = 1, \quad \therefore \alpha = \beta; \quad 2\cos i = \gamma, \quad \therefore \cos i = \tfrac{1}{2}\gamma:$$

from the two first there results the condition, that the coefficients of the squares in the equation must either be 1 and positive, or they must be equal in magnitude and sign; from the second, that the cosine of inclination of the axes is equal to half the coefficient of the rectangle $xy$. Now, since the greatest value of a cosine is $\pm 1$, it follows as another condition, that $\gamma$ must not be $> \pm 2$. When, therefore, $\alpha$ and $\beta$ are equal in magnitude and sign, and $\gamma$ not $> \pm 2$, the general equation (G) always represents a circle, whatever real values $\delta$, $\epsilon$, $\zeta$, may have.

Ex. Required to find which of the following equations represent a circle:

$$x(x - 6) = y(y - x) + 5(y + 2),$$
$$x^2 + y^2 - 5(y + 3) = 3x(2 - y),$$
$$4y(y - 1) - 2x(1 - 2x) = 6(3 - xy),$$
$$x(x - 1) + 2\{x - y(1 - y)\} + 2(x + 3) = 0;$$

multiplying out, and transposing the important terms to the left-hand side, the equations become

$$x^2 - y^2 + xy + \&c. = 0 \ldots\ldots\ldots\ldots (1).$$
$$x^2 + y^2 + 3xy + \&c. = 0 \ldots\ldots\ldots\ldots (2).$$

$$4x^2 + 4y^2 + 6xy + \&c. = 0 \ldots\ldots\ldots\ldots (3).$$
$$x^2 + 2y^2 + xy + \&c. = 0 \ldots\ldots\ldots\ldots (4).$$

(1) cannot represent a circle, as the signs of the squares are different, although satisfying the other conditions.

(2) cannot represent a circle, because the coefficient of $xy$ is $> 2$.

(3) when divided by 4, so as to have 1 for the coefficient of each square, becomes

$$x^2 + y^2 + \tfrac{6}{4}\,xy + \&c. = 0,$$

and $\frac{6}{4}$ being $< 2$, the three conditions are satisfied, therefore the locus of that equation is a circle.

(4) cannot represent a circle, because the coefficients of the squares are not equal.

6. To trace the circle whose equation is given, referred to oblique axes.

The process is the same in this case as when the axes are rectangular; but besides $a$, $b$, and $r$, the angle at which the co-ordinate axes are inclined must also be determined by means of the coefficient of $xy$. Hence, as a rule, put the given equation under the form

$$x^2 + y^2 + Axy + Bx + Cy + D = 0.$$

Arrange the terms in the equation to the circle in the same order, and equate to $A$, $B$, $C$, $D$, those quantities in it which correspond to them.

Ex. Let the given equation be

$$x^2 + y^2 + xy - 14x - 13y + 25 = 0.$$

The equation to the circle, under the same form, is

$$x^2 + y^2 + 2xy \cos i - 2(a + b \cos i)\,x - 2(b + a \cos i)\,y$$
$$+ a^2 + b^2 + 2ab \cos i - r^2 = 0.$$

Hence, equating coefficients of like terms,

$$2 \cos i = 1 \ldots\ldots\ldots\ldots (1),$$
$$2(a + b \cos i) = 14 \ldots\ldots\ldots\ldots (2),$$
$$2(b + a \cos i) = 13 \ldots\ldots\ldots\ldots (3),$$
$$a^2 + b^2 + 2ab \cos i - r^2 = 25 \ldots\ldots\ldots\ldots (4).$$

From (1), $\cos i = \frac{1}{2}$, and by Trigonometry, $\cos 60° = \frac{1}{2}$,

$$\therefore i = 60°.$$

Substituting (1) in (2) and (3), these become

$$2a + b = 14,$$
$$2b + a = 13,$$

which solved gives $a = 5, \quad b = 4$;

and substituting in (4),

$$25 + 16 + 20 - 25 = r^2, \quad \therefore r^2 = 36, \quad r = 6.$$

Hence, if $OX$, $OY$, (fig. 87) be drawn inclined at 60°, and $OA$ be taken $= 5$, $AC$ parallel to $OY = 4$, and $CP = 6$; the circle described from $C$, with $CP$ as the radius, will be the locus of the given equation.

When (1) gives for $\cos i$ a fraction from which the angle cannot be readily determined, the student must construct from the cosine as follows:

Required the axes inclined at $\cos^{-1}(\frac{2}{7})$, $\cos^{-1}(\frac{5}{7})$, $\cos^{-1}(\frac{-3}{7})$.

Take a line to determine the axis of $x$, and divided into as many parts as there are units in the denominator, and with it as radius describe a circle: from the centre take along the axis of $x$ a distance corresponding to the units in the numerator, *attending to the sign*, and from the point so determined draw a perpendicular to cut the circle; then the radius drawn to the point of intersection will determine the axis of $y$.

Let $OR$ (fig. 88) coincide with the axis of $x$ and $= 7$; take $ON_1 = 2$, $ON_2 = 5$, and $ON_3$ on the negative side $= 3$, and draw the perpendiculars $N_1P_1$, $N_2P_2$, $N_3P_3$; then from the triangle $P_1ON_1$,

$$\cos P_1ON_1 = \frac{ON_1}{OP_1} = \frac{2}{7}, \quad \therefore P_1ON_1 = \cos^{-1}\left(\frac{2}{7}\right);$$

and $P_1O$, $OR$, produced indefinitely, will be the axes inclined at $\cos^{-1}(\frac{2}{7})$; and similarly

$OP_2$, $OR$, will be inclined at $\cos^{-1}(\frac{5}{7})$,

$OP_3$, $OR$, ... ... ... ... $\cos^{-1}(\frac{-3}{7})$.

7. To determine the points in which a straight line intersects a circle.

The principle in this case is the same as when two lines intersect, viz. the values of the co-ordinates of the point of intersection must be such as to satisfy both the equation to the line and that to the circle, or more generally: *when two loci intersect the co-ordinates (referred to the same axes) of the point of intersection satisfy the equation to each locus.* Hence two equations are obtained, one from each locus, and by combining them the values of the co-ordinates are ascertained.

For brevity, assume the centre of the circle as the origin, and the equations to the circle and line to be referred to rectangular axes: they will be

$$x^2 + y^2 = r^2 \ldots\ldots\ldots\ldots (1),$$

$$y = tx + c \ldots\ldots\ldots (2).$$

Substituting the value of $x$ obtained from (2) into (1), the result is

$$r^2 = \left(\frac{y - c}{t}\right)^2 + y^2;$$

expanding, transposing, and completing the square,

$$y^2 - \frac{2c}{1 + t^2}\,y + \frac{c^2}{(1 + t^2)^2} = \frac{r^2t^2 - c^2}{1 + t^2} + \frac{c^2}{(1 + t^2)^2};$$

from which is obtained

$$y = \frac{c \pm \sqrt{\{r^2(1 + t^2) - c^2\}}}{1 + t^2}.$$

This equation gives for $y$ either two impossible or two real and unequal values, or one real value. The values are impossible, or the line does not cut the circle if the quantity under the radical sign be negative; that is, if

$$c^2 > r^2(1 + t^2), \quad \text{or} \quad c > r\sqrt{(1 + t^2)},$$

that is, if the radius be less than the projection of $c$ upon it.

If the quantity under the radical sign be positive, then $y$ has two values, or the line cuts the circle in two points. If the quantity under the radical sign $= 0$, then $y$ has only one value, or the line touches the circle, but does not cut it.

The student will observe, that these conditions hold equally when $t$ and $c$ are affected with the negative sign; for both being involved to the square in the expression under the radical, the negative sign does not affect them.

8. To find the co-ordinates of the points in which two circles intersect.

Let the origin be the centre of one of the circles, and let $(a, b)$ be the co-ordinates of the centre of the other, $r_1$, $r_2$, being their radii respectively; then, since the co-ordinates of the points

of intersection must satisfy the equation to each circle, if $(x'y')$ be a point of intersection,

$$x'^2 + y'^2 = r_1^2 \ldots\ldots\ldots\ldots (1),$$

$$(x' - a)^2 + (y' - b)^2 = r_2^2 \ldots\ldots\ldots\ldots (2).$$

Expanding (2),

$$x'^2 - 2x'a + a^2 + y'^2 - 2y'b + b^2 = r_2^2;$$

or since $\quad x'^2 + y'^2 = r_1^2$, from (1);

$$\therefore\ x' = \frac{1}{2a}\{a^2 + b^2 - (r_1^2 - r_2^2) - 2by'\} \ldots (3).$$

This value of $x'$ substituted in (1) would give a quadratic for finding the values of $y'$; and substituting these in (3), the corresponding values of $x'$ would be known.

To make the operation less troublesome, let the centre of the second circle be in the axis of $x$, i.e. let its co-ordinates be $(a, o)$; then

$$x'^2 + y'^2 = r_1^2 \ldots\ldots\ldots (1),$$

$$(x' - a)^2 + y'^2 = r_2^2 \ldots\ldots\ldots (2).$$

Expanding (2), $\quad x'^2 - 2x'a + a^2 + y'^2 = r_2^2,$

or $\quad -2x'a + a^2 = r_2^2 - r_1^2;$

$$\therefore\ x' = \frac{1}{2a}(a^2 + r_1^2 - r_2^2).$$

Substituting this value in (1), and transposing,

$$y'^2 = r_1^2 - \frac{(a^2 + r_1^2 - r_2^2)^2}{4a^2};$$

and treating this by the formula $A^2 - B^2 = (A + B)(A - B)$,

$$y'^2 = \left\{r_1 + \frac{(a^2 + r_1^2 - r_2^2)}{2a}\right\} \cdot \left\{r_1 - \frac{(a^2 + r_1^2 - r_2^2)}{2a}\right\}$$

$$= \frac{(2ar_1 + a^2 + r_1^2 - r_2^2)}{2a} \cdot \frac{(2ar_1 - a^2 - r_1^2 + r_2^2)}{2a}$$

$$= \frac{\{(a + r_1)^2 - r_2^2\}}{2a} \cdot \frac{\{r_2^2 - (a - r_1)^2\}}{2a};$$

$$\therefore\ y' = \pm\frac{1}{2a}\sqrt{\{(a + r_1 + r_2)(a + r_1 - r_2)(r_2 - a + r_1)(r_2 + a - r_1)\}}.$$

In these results there are two values of $y$ to one of $x$, which ought to be the case, since the axis of $x$ passing through both

centres must be perpendicular to the chord passing through the points of intersection, as the student knows from Geometry, and therefore for every point in the chord $x'$ has the same value.

*Tangent and Normal to a Circle.*

9. Def. The tangent to a circle is a line which meets the circle, and being produced does not cut it.

The point in which the tangent meets the circle is called the point of contact.

The normal to a circle, or to any curve, is a line perpendicular to the tangent to the circle or curve at the point of contact.

Since then the normal and tangent are two lines perpendicular to each other, the following relation (as the student has seen in the last chapter) always obtains between them:

if one be inclined to the axis of $x$ at $\tan^{-1}(t)$,

the other is .................... $\tan^{-1}\left(-\frac{1}{t}\right)$;

and therefore, knowing the equations to either of these lines, that to the other may be immediately obtained.

Again, the student knows from (*Euclid*, 18. 3), that "If a right line touch a circle, the right line drawn from the centre to the point of contact shall be perpendicular to the line which touches the circle:" *i.e.* the radius is a normal to the circle,* or *a normal to the circle always passes through its centre.*

For convenience of investigation the centre of the circle is taken for the origin of the co-ordinates.

10. To find the equation to the normal to a circle.

Let the centre of the circle be taken for the origin of the co-ordinates, and let $(\alpha, \beta)$ be the point in the circumference through which the normal passes. Since, by the property of the circle, the radius drawn to $(\alpha, \beta)$ is a normal to the circle, conversely the normal at that point passes through the centre; and as this is assumed as the origin of the co-ordinates, the equation to the normal will be of the form

$$y = tx \text{ .................. } (1);$$

* See Properties of the Circle, 6*.

also, since it passes through $(\alpha, \beta)$, the values of $\alpha$ and $\beta$ must be such as to satisfy (1),

$$\therefore \beta = t\alpha, \quad \therefore t = \frac{\beta}{\alpha} \ldots\ldots\ldots\ldots (2).$$

Substituting (2) in (1),

$$y = \frac{\beta}{\alpha} x \ldots\ldots\ldots\ldots\ldots\ldots (N),$$

which is the equation to the normal, or to the radius drawn to $(\alpha, \beta)$.

11. To find the equation to the tangent to a circle.

Let the centre of the circle be taken for the origin, and let $(\alpha, \beta)$ be the point of contact.

Now, the tangent being a line passing through $(\alpha, \beta)$, its equation will be of the form

$$\frac{y-\beta}{x-\alpha} = t \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

By the property of the circle, the tangent at any point is perpendicular to the radius drawn to that point;* but the equation to a radius drawn to $(\alpha, \beta)$ is

$$y = \frac{\beta}{\alpha} x \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Hence, by the relation between perpendicular lines,

$$t = -\frac{\alpha}{\beta} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Substituting (3) in (1), the equation to the tangent is

$$y - \beta = -\frac{\alpha}{\beta}(x - \alpha) \ldots\ldots\ldots\ldots (T).$$

COR. 1. This may be put into another form: multiplying and transposing,

$$y\beta - \beta^2 = -\alpha x + \alpha^2,$$
$$\therefore y\beta + \alpha x = \alpha^2 + \beta^2.$$

But, since $(\alpha, \beta)$ is a point in the circumference, if $r$ be the radius, the equation to the circle gives

$$\beta^2 + \alpha^2 = r^2.$$

Hence, substituting in the preceding equation, it becomes

$$y\beta + x\alpha = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (T_2).$$

COR. 2. Let $(h, k)$ be another point through which the tangent passes (which of course cannot be a point in the circum-

* See Properties of the Circle, 6*.

ference, since, by the definition, the tangent only meets it in one point); then, since the co-ordinates of all points in a line satisfy its equation, substituting in $(T_2)$,

$$h\alpha + k\beta = r^2 \ldots\ldots\ldots\ldots\ldots (T_3).$$

When a tangent is to be drawn from a given point $(h, k)$, $(T_3)$ expresses the equation of condition which $h$ and $k$ must satisfy, in order that it may touch the circle at the point $(\alpha, \beta)$.

COR. 3. And it follows, also, that the condition is equally binding upon the values of $\alpha$ and $\beta$, and therefore if $(x, y)$ be an unknown point of contact, to which a tangent is drawn from $(h, k)$, the values of $x$ and $y$ must satisfy the equation

$$hx + ky = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (T_4),$$

as well as the equation to the circle.

12. To determine the points of contact, when tangents are drawn from a given point to a given circle.

Let $(h, k)$ be the given point referred to the centre of the circle as origin, let $r$ be the radius of the circle, and $(x, y)$ a point of contact to be determined.

The values of $x$ and $y$ must satisfy the equations to the tangent and to the circle. Hence

$$hx + ky = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x^2 + y^2 = r^2 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\text{from (1),} \quad x = \frac{r^2 - ky}{h} \ldots\ldots\ldots\ldots (3).$$

Substituting (3) in (2),

$$y^2 + \left(\frac{r^2 - ky}{h}\right)^2 = r^2 \ldots\ldots\ldots\ldots (4).$$

Solving the quadratic (4), two values of $y$ are obtained, which being substituted in (3), will give the corresponding values of $x$; hence, it follows that from a given point two tangents may be drawn to a circle, as the student knows from (Euclid 3, 8).

13. The equation to the secant passing through the two points to which tangents may be drawn from a given point $(h, k)$, is the same as $(T_4)$, the equation to the tangent when the point of contact is indeterminate.

G

Let $\tan^{-1} t$ be the inclination of the secant to the axis of $x$, $(x_1, y_1)$ one of the points of contact. Since it passes through $(x_1, y_1)$ its equation is

$$y - y_1 = t(x - x_1) \ldots\ldots\ldots\ldots\ldots (1).$$

Now, by a property of the circle, the line drawn from the centre to the middle point of the chord (the part of the secant within the circle) will be perpendicular to the secant,* and if produced, will pass through the point $(h, k)$, from the known property that the tangents are equal.† Hence, its equation will be

$$y = \frac{k}{h} x \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since, then, (2) and (1) are lines perpendicular to each other,

$$t = -\frac{h}{k}.$$

Substituting in (1), $\quad y - y_1 = -\frac{h}{k}(x - x_1)$;

multiplying and transposing, this becomes

$$hx + ky = hx_1 + ky_1,$$

but $\quad hx_1 + ky_1 = r^2 \quad$ from equation to tangent,

$$\therefore \quad hx + ky = r^2$$

is the equation to the secant.

14. To construct or trace the tangents to a circle from a given point $(h, k)$.

What is required here is to determine the points of contact. In (12) it was shewn how this might be done analytically, but the property of the secant just proved suggests an easy geometrical method. For, since that line passes through the points of contact, if it be constructed from its equation, the points in which it intersects the circle will be the points required.

$x$ and $y$ being the co-ordinates of any point in the secant, its equation is $\qquad hx + ky = r^2$;

to construct which, the points in which it cuts the axes of $x$ and of $y$ must be determined.

Putting $y = 0$, $\qquad x = \frac{r^2}{h}$,

.... $\quad x = 0$, $\qquad y = \frac{r^2}{k}$;

* See Properties of the Circle 3*. † Ibid. 7*.

hence, (fig. 89) if $OH$ be taken $= \frac{r^2}{h}$,

.............. $OK$ ........ $= \frac{r^2}{k}$;

the line $HR$ will intersect the circle in the points $P$ and $Q$, and these are the required points of contact.

To find $OH$, $OK$ by a geometrical construction. Since

$$OH = \frac{r^2}{h} \quad \therefore \quad \frac{OH}{r} = \frac{r}{h} \quad \therefore \quad \frac{h}{r} = \frac{r}{OH};$$

take (fig. 90) $AB$, $AC$, lines of indefinite length,

$$AM = h, \quad MR = r, \quad AN = r;$$

join $MN$, and draw $RS$ parallel to $MN$; then, by similar triangles,

$$\frac{AM}{MR} = \frac{AN}{NS}, \quad \text{or} \quad NS = \frac{r^2}{h} = OH;$$

in the same manner $OK$ may be determined.

15. If a line be drawn perpendicular to the axis of $x$, and from different points in it pairs of tangents be drawn to the circle, the secants passing through the points of contact will all cut the axis of $x$ in the same point.

Let a line be drawn perpendicular to the axis of $x$ at a point $h$ in that axis, then $(h, y_1)$ $(h, y_2)$ will be the co-ordinates of two points in it. Conceive pairs of tangents to be drawn from these points, then the equations to the secant passing through these points of contact will be

$$xh + yy_1 = r^2 \text{..................(1)},$$
$$xh + yy_2 = r^2 \text{..................(2)}.$$

Putting $y = 0$,

(1) cuts the axis of $x$ in a point $x = \frac{r^2}{h}$,

(2) ...................... $x = \frac{r^2}{h}$;

that is, they both pass through the same point in the axis of $x$.

The same might be proved of any other points in the perpendicular line.

Cor. As the position of the co-ordinate axes is quite arbitrary, if any other line were proposed, the axis of $x$ might be chosen

perpendicular to it, and the preceding investigation would apply; hence, generally, when tangents are drawn to a circle from different points in a straight line, the secants through the points of contact cut the perpendicular line through the centre in the same point.

16. To find the equation to the tangent by Descartes' method.

Let (fig. 91) the centre of the circle be taken for origin; draw a secant $LPQl$ through any two points $P$, $Q$, and let $(x_1, y_1)$ be the co-ordinates of $P$, $(x_2, y_2)$ be the co-ordinates of $Q$.

The equation to $LPQl$, because it passes through $(x_1, y_1)$ $(x_2, y_2)$, is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) \ldots\ldots\ldots\ldots (1).$$

But, since $(x_1, y_1)$ $(x_2, y_2)$ are points in a circle, they must satisfy the equation to the circle. Hence

$$x_1^2 + y_1^2 = r^2,$$
$$x_2^2 + y_2^2 = r^2;$$

subtracting and transposing,

$$y_1^2 - y_2^2 = -(x_1^2 - x_2^2),$$
$$\therefore (y_1 - y_2)(y_1 + y_2) = -(x_1 - x_2)(x_1 + x_2),$$
$$\therefore \frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2}{y_1 + y_2}.$$

Substituting (1) becomes

$$y - y_1 = -\frac{x_1 + x_2}{y_1 + y_2}(x - x_1) \ldots\ldots\ldots\ldots (2).$$

Conceive the secant to be made to revolve about the point $P(x_1, y_1)$, then the second point of intersection will approach nearer and nearer to $P$, and at last coincide with it, in which case

$$y_2 \text{ will} = y_1,$$
$$x_2 \;..\; = x_1;$$

$$\text{and } \frac{x_1 + x_2}{y_1 + y_2} \text{ will} = \frac{2x_1}{2y_1};$$

substituting this value in (2), it becomes

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1) \ldots\ldots\ldots\ldots (3);$$

multiplying up and transposing,

$$yy_1 + xx_1 = y_1^2 + x_1^2;$$

but because $(x_1, y_1)$ is a point in the circle,

$$y_1^2 + x_1^2 = r^2,$$

therefore, substituting in (3),

$$yy_1 + xx_1 = r^2,$$

which is the equation obtained before.

This method of obtaining the equation to the tangent by the revolution of a secant, the student must impress on his memory, as it is applicable to other curves.

## *Properties of the Circle.*

The student having proceeded thus far, may perhaps take it into his head to count his gains, and if he has found difficulty in understanding what he has read, will possibly condemn the reading as *dry*, if not *unprofitable*, having as yet only learned to represent unknown curves, a line, and a circle, by means of letters. Perhaps he may exclaim—What is the use! The best way to reply to this burst of indignation seems to be, not to promise him wonders when he comes to Dynamics and Physical Astronomy, but to shew him the services which analysis can render him, even now, in cases which he can appreciate. With this object in view, the following propositions have been prepared, and it is thought that he will be fully inclined to give analysis credit for its utility, when he sees some of the more important results in Euclid's Third Book arrived at by its operations and principles, without borrowing from Geometry even the figure of a circle.

1*. THE centre of the circle is the origin of co-ordinates, and the equation to a line is $x = a$; to determine the circumstances of its intersection with the circle, when the length of the radius is $r$.

Since the centre is the origin, the equation to the circle is

$$x^2 + y^2 = r^2.$$

And, supposing the line $(x = a)$ to intersect it, the same value of $x$ must satisfy both equations. Hence

$$a^2 + y^2 = r^2 \quad \therefore\ y = \pm \sqrt{(r^2 - a^2)}.$$

First, this equation is always possible so long as $a < r$; *i. e.* that the given line, which by its form is perpendicular to the

axis of $x$, may intersect the circle, the point $(a, o)$ in which it cuts the axis of $x$, must be within the circle.

Secondly, when this is the case, $y$ has two equal values with different signs. Let $\beta$, $-\beta$, be these values; then, as the given line is perpendicular to the axis of $x$, $a$ will be the abscissa to both points; it therefore intersects the circle in $(a, \beta)$ $(a, -\beta)$, which from the sign of $\beta$ fall on opposite sides of the axis of $x$.

The straight line included between the points $(a, \beta)$ $(a, -\beta)$ is called a chord. Let $C$ be its length; then, from formula ($\delta$),

$$C = \sqrt{\{(a-a)^2 + (\beta+\beta)^2\}} = 2\beta.$$

2*. A line drawn from the centre perpendicular to a chord bisects it. (Euclid 3, 3.)

Let $(x_1, y_1)$ $(x_2, y_2)$ be the co-ordinates of the extremities of the chord, and let $y_1 > y_2$. Let $\theta$ be its inclination to the axis of $x$, then

$$\tan\theta = \frac{y_1 - y_2}{x_1 - x_2};$$

and since the chord passes through $(x_1, y_1)$, its equation is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) \ldots\ldots\ldots\ldots (1);$$

from the equation to the circle,

$$y_1^2 + x_1^2 = r^2,$$

$$y_2^2 + x_2^2 = r^2;$$

$$\therefore\ y_1^2 - y_2^2 = -(x_1^2 - x_2^2),$$

$$\therefore\ \frac{y_1 - y_2}{x_1 - x_2} = -\frac{x_1 + x_2}{y_1 + y_2} \ldots\ldots\ldots\ldots (2).$$

Substituting (2) in (1), the equation to the chord is

$$y - y_1 = -\frac{x_1 + x_2}{y_1 + y_2}(x - x_1) \ldots\ldots\ldots\ldots (3).$$

Let $(X, Y)$ be the co-ordinates of the point in which the perpendicular from the centre on the chord intersects the chord; then from (3), its equation, since it passes through the centre (the origin), will give

$$Y = \frac{y_1 + y_2}{x_1 + x_2} X \ldots\ldots\ldots\ldots (4).$$

Since (4) and (3) intersect, the same value of $x$ and $y$ must satisfy both equations. Hence, equating the values of $Y$,

$$\frac{y_1 + y_2}{x_1 + x_2} X = y_1 - \frac{x_1 + x_2}{y_1 + y_2}(X - x_1);$$

$$\therefore X\left(\frac{y_1 + y_2}{x_1 + x_2} + \frac{x_1 + x_2}{y_1 + y_2}\right) = y_1 + \frac{x_1 + x_2}{y_1 + y_2} x_1;$$

multiplying up and substituting for $(x_1^2 + y_1^2)(x_2^2 + y_2^2)$ from the equation to the circle, it will be found that

$$X = \tfrac{1}{2}(x_1 + x_2)\ldots\ldots\ldots\ldots\ldots(5);$$

and substituting in (4),

$$Y = \tfrac{1}{2}(y_1 + y_2)\ldots\ldots\ldots\ldots\ldots(6);$$

and the point of intersection is determined.

Now, the point $(X, Y)$ is within the circle; if not it either is a point in the circumference or without it, in which case

$$X^2 + Y^2 \text{ must } = \text{ or } > r^2 \ldots. \text{(A)},$$

$$\therefore (x_1 + x_2)^2 + (y_1 + y_2)^2 \ldots. = \text{ or } > 4r^2,$$

$$\therefore x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2x_1x_2 + 2y_1y_2 = \text{ or } > 4r^2,$$

$$\therefore x_1x_2 + y_1y_2 = \text{ or } > r^2,$$

$$\therefore \frac{x_1}{r}\frac{x_2}{r} + \frac{y_1}{r}\frac{y_2}{r} = \text{ or } > 1.$$

Now from the equation to the circle

$$\frac{x_1}{r} = \sqrt{\left(1 - \frac{y_1^2}{r^2}\right)}, \quad \frac{x_2}{r} = \sqrt{\left(1 - \frac{y_2^2}{r^2}\right)}.$$

Hence, if (A) holds,

$$\sqrt{\left(1 - \frac{y_1^2}{r^2}\right)} \cdot \sqrt{\left(1 - \frac{y_2^2}{r^2}\right)} \text{ must} = \text{ or } > 1 - \frac{y_1}{r}\frac{y_2}{r},$$

$$\therefore \left(1 - \frac{y_1^2}{r^2}\right)\left(1 - \frac{y_2^2}{r^2}\right)\ldots. = \text{ or } > 1 + \frac{y_1^2}{r^2}\frac{y_2^2}{r^2} - \frac{2y_1y_2}{r.r},$$

$$\therefore 1 + \frac{y_1^2}{r^2}\frac{y_2^2}{r^2} - \left(\frac{y_1^2 + y_2^2}{r^2}\right)\ldots. = \text{ or } > 1 + \frac{y_1^2}{r^2}\frac{y_2^2}{r^2} - \frac{2y_1y_2}{r.r},$$

$$\therefore -\frac{y_1^2 + y_2^2}{r^2}\ldots. = \text{ or } > -\frac{2y_1y_2}{r.r},$$

$$\therefore \frac{2y_1y_2}{r.r}\ldots. = \text{ or } > \frac{y_1^2 + y_2^2}{r^2} \ldots \text{(B)}.$$

But since $y_1$, $y_2$ are unequal by hypothesis,

$$2y_1y_2 \text{ must be } < y_1^2 + y_2^2,$$

as the student knows from the formula

$$(y_1 - y_2)^2 = \text{a positive quantity};$$

hence (B) is not satisfied: therefore (A), from which (B) is derived, is not satisfied, or the point $(X, Y)$ is within the circle.

Let $d$ be its distance (the length of the perpendicular) from the centre, and let $c_1$, $c_2$ be the parts into which it divides the chord,

$c_1$ being the part included between $(X, Y)$ $(x_1, y_1)$,
$c_2$ ........................ $(X, Y)$ $(x_2, y_2)$;

then, since the chord is perpendicular to $d$, and its extremities are in the circumference

$$r^2 = d^2 + c_1^2,$$
$$r^2 = d^2 + c_2^2;$$
$$\therefore c_1 = c_2,$$

or the chord is bisected by the perpendicular upon it from the centre.

3*. If a straight line, drawn through the centre of a circle, bisect a straight line in it which does not pass through the centre, it shall cut it at right angles. (Euclid 3, 3.)

Let $(x_1, y_1)$ $(x_2, y_2)$ be the extremities of the chord,
$(X, Y)$ ........ the middle point..........
$c_1$ the part included between $(x_1, y_1)$ and $(X, Y)$,
$c_2$ ........................ $(X, Y)$ and $(x_2, y_2)$.

Now the equation to the chord is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) \text{............} (1);$$

and the equation to the line drawn from the centre through $(X, Y)$ is

$$y = \frac{Y}{X}x \text{.................} (2);$$

and, by formula ($\delta$),

$$c_1 = \sqrt{\{(y_1 - Y)^2 + (x_1 - X)^2\}},$$
$$c_2 = \sqrt{\{(Y - y_2)^2 + (X - x_2)^2\}};$$

and since these are equal, for by hypothesis the chord is bisected,

$$(y_1 - Y)^2 + (x_1 - X)^2 = (Y - y_2)^2 + (X - x_2)^2,$$

$$\therefore y_1^2 - 2y_1 Y + Y^2 + x_1^2 - 2x_1 X + X^2 = Y^2 - 2y_2 Y + y_2^2 + X^2 - 2x_2 X + X_2^2,$$

$$\therefore y_1^2 + x_1^2 - 2y_1 Y - 2x_1 X = y_2^2 + x_2^2 - 2y_2 Y - 2x_2 X;$$

but $\because$ $(x_1, y_1)$ $(x_2, y_2)$ are in the circumference

$$y_1^2 + x_1^2 = r^2 = y_2^2 + x_2^2;$$

$$\therefore 2y_1 Y + 2x_1 X = 2y_2 Y + 2x_2 X,$$

$$\therefore Y(y_1 - y_2) = -X(x_1 - x_2),$$

$$\therefore \frac{Y}{X} = -\frac{x_1 - x_2}{y_1 - y_2} \ldots\ldots\ldots\ldots\ldots (3).$$

Hence, the equation to the line drawn to $(X, Y)$ is

$$y = -\frac{x_1 - x_2}{y_1 - y_2} x \ldots\ldots\ldots\ldots\ldots (4);$$

and (4) and (1) satisfy the conditions of lines perpendicular to each other, therefore a line drawn from the centre to the middle point of a chord is perpendicular to the chord.

4*. If any two points be taken in the circumference of a circle, the straight line that joins them shall fall within the circle. (Euclid 3, 2.)

Let the axis of $x$ be chosen perpendicular to the line drawn between the two points; then, as was seen in the last proposition, the chord will be bisected, and its extremities will fall on the opposite sides of the axis of $x$. Let $(a, \beta)$ $(a, -\beta)$ be its extremities, and take any point in it $(a, b)$; let $d$ be the distance of this point from the centre, then

$$d^2 = a^2 + b^2;$$

also, from the equation to the circle,

$$r^2 = a^2 + \beta^2,$$

$$\therefore r^2 - d^2 = \beta^2 - b^2:$$

but since the chord cannot extend beyond its extremities $(a, \beta)$ $(a, -\beta)$, $b$, whether positive or negative, cannot be $> \beta$; it must therefore be less, if $(a, b)$ be a point different from $(a, \beta)$ or $(a, -\beta)$. Hence

$$r^2 - d^2 \text{ is positive, or } r^2 > d^2, \quad \therefore r > d,$$

and the point $(a, b)$ is within the circle.

5*. In a circle the angle at the centre is double the angle at the circumference, upon the same base that is upon the same part of the circumference. (Euclid 20, 3.)

Let the centre of the circle be the origin, then $(r, o)$ $(-r, o)$ will be the extremities of the diameter coinciding with the axis of $x$.

Let one extremity of the arc subtending the angles coincide with $(r, o)$, and let $(x, y)$ be the other extremity; then the equation to a line from the centre through $(x, y)$ will be

$$y = tx \quad \dots\dots \quad (1),$$

and the equation to a line from $(-r, o)$ to $(x, y)$ will be

$$y = t'(x + r) \quad \dots\dots \quad (2).$$

The equation to the circle is

$$x^2 + y^2 = r^2 \quad \dots\dots \quad (3).$$

Substituting from (3) into (2),

$$\begin{aligned} y &= t'\{x + \sqrt{(x^2 + y^2)}\}, \\ &= t'x\left\{1 + \sqrt{\left(1 + \frac{y^2}{x^2}\right)}\right\} \\ &= t'x\{1 + \sqrt{(1 + t^2)}\} \quad \dots\dots \quad (4), \end{aligned}$$

by substituting from (1).

Since the lines from the centre and $(-r, o)$ intersect in $(x, y)$, their equations must be satisfied by the same values of $x$ and $y$. Hence, equating (4) and (1),

$$\begin{aligned} tx &= t'x\{1 + \sqrt{(1 + t^2)}\}, \\ \therefore (t - t')^2 &= t'^2(1 + t^2), \\ \therefore t^2 - 2t't + t'^2 &= t'^2 + t'^2t^2, \\ \therefore t^2 - 2t't &= t'^2t^2, \\ \therefore t - t'^2t &= 2t', \\ \therefore t &= \frac{2t'}{1 - t'^2}. \end{aligned}$$

Comparing this result with the Trigonometrical formula, it appears that $\tan^{-1}t = 2\tan^{-1}t'$,

*i.e.* angle at centre = 2 angle at circumference.

6*. A tangent to a circle is perpendicular to the radius.

Let the line whose equation is $x = a$ intersect the circle; then,

as was shewn in the first proposition, $r > a$ and the ordinates of the points of intersection are obtained from the equation

$$y = \pm \sqrt{(r^2 - a^2)} \dots\dots\dots\dots\dots\dots (1).$$

Let $r - a = nh$; and conceive lines parallel to the line $(x = a)$ to be drawn through the points $(a + h, o)$ $(a + 2h, o)$...$(a + nh, o)$; the values of the ordinates of the points of intersection will be obtained from the equations

$$y_1 = \pm \sqrt{\{r^2 - (a + h)^2\}},$$
$$y_2 = \pm \sqrt{\{r^2 - (a + 2h)^2\}},$$
$$\dots\dots = \dots\dots\dots\dots\dots\dots$$
$$y_{n-1} = \pm \sqrt{[r^2 - \{a + (n - 1)\, h\}^2]},$$
$$y_n = \pm \sqrt{\{r^2 - (a + nh)^2\}}.$$

But, since by hypothesis $a + nh = r$,

$$y_n = \pm \sqrt{(r^2 - r^2)} = \pm\, 0 = 0.$$

It appears, then, that $y_1\, y_2 \dots y_{n-1}$ have each *two* equal values with different signs, and therefore that each of the lines drawn through the points $(a + h, o)$ $(a + 2h, o) \dots \{a + (n - 1)\, h, o\}$, meets the circumference in *two points*, whose co-ordinates are

$$(a + h, y_1)\,(a + h, - y_1),\ (a + 2h, y_2)\,(a + 2h, - y_2), \dots\dots$$

respectively; but that the line through $(a + nh, o)$ meets the circle in *one point only*, viz. $(a + nh, o)$ or $(r, o)$; hence, by definition, this line is a tangent; but by hypothesis it is parallel to the line $(x = a)$, which by the form of its equation must be perpendicular to the axis of $x$; and the point $(r, o)$ is the extremity of a radius coinciding with the axis of $x$; hence, the tangent through $(r, o)$ is perpendicular to the radius drawn to that point. And since the position of the axis of $x$ is quite arbitrary, it might be assumed to pass through *any other* point in the circumference, and the same investigation would apply; therefore, generally, the tangent at the extremity of a radius is perpendicular to the radius.

7*. If two tangents be drawn at the extremities of a chord to intersect, the portions of them between the circumference and point of intersection are equal.

Let the axis of $x$ be so assumed that the chord may be perpendicular to it, and let $x = a$ be the equation to the chord, then the co-ordinates of its extremities will be $(a, \beta)$ and $(a, - \beta)$.

Now, $(x', y')$ being a point of contact, the equation to a tangent is

$$yy' + xx' = r^2.$$

Hence, the equation to a tangent at the points $(\alpha, \beta)$ $(\alpha, -\beta)$ will be

$$y\beta + x\alpha = r^2 \ldots\ldots\ldots\ldots\ldots (1),$$

$$-y\beta + x\alpha = r^2 \ldots\ldots\ldots\ldots\ldots (2).$$

Putting $y = 0$ to find where they intersect the axis of $x$,

$$x = \frac{r^2}{\alpha}$$

is the value obtained from (1) and (2). Hence, when a chord of a circle is perpendicular to an axis, the tangents at its extremities intersect that axis in the same point. And, as one of the axes may always be chosen so as to coincide with any line passing through the centre, it follows generally that the tangents at the extremity of a chord of a circle, intersect a line from the centre perpendicular to that chord in the same point.

The extremities of the tangents then are $(\alpha, \beta)$, $\left(\frac{r^2}{\alpha}, 0\right)$, $(\alpha, -\beta)$, $\left(\frac{r^2}{\alpha}, 0\right)$. Let $T_1$, $T_2$, be their lengths; then, by formula ($\delta$)

$$T_1 = \sqrt{\left\{\left(\alpha - \frac{r^2}{\alpha}\right)^2 + \beta^2\right\}},$$

$$T_2 = \sqrt{\left\{\left(\alpha - \frac{r^2}{\alpha}\right)^2 + \beta^2\right\}},$$

$$\therefore\ T_1 = T_2,$$

or the tangents are equal.

8*. If two lines within a circle cut one another, the rectangle contained by the segments of one of them is equal to the rectangle contained by the segments of the other. (*Euclid*, 35, 3.)

Let $r$ be the radius of the circle, and the centre the origin; $d$ the distance of the point of intersection from the centre; and let the axes be so chosen that the axis of $x$ may be perpendicular to one of the lines, the extremities of this line when referred to these axes being $(\alpha, \beta)$, $(\alpha, -\beta)$.

Then, since the point of intersection is a point in this line, it will have the same abscissa as its extremities; let $b$ be its

ordinate, *i.e.* let $(a, b)$ be the point whose distance from the origin is $d$; then

$$d^2 = a^2 + b^2 \ldots\ldots$$

$$\text{also} \quad r^2 = a^2 + \beta^2 \ldots\ldots$$

$$\therefore r^2 - d^2 = \beta^2 - b^2,$$

$$(r + d)(r - d) = (\beta - b)(\beta + b) \ldots\ldots$$

Now the distance of the point of intersection from the extremities of the chord will, by formula ($\delta$), be found to be $(\beta - b)$ and $(\beta + b)$. Hence, calling these $p$, $P$, respectively,

$$(r + d)(r - d) = p \, . \, P.$$

The product on the left-hand side involving only $r$ and $d$, which are distances from the centre, is independent of the position of the axes; and so, in fact, is the absolute length of the parts $p$ and $P$. Hence, if $q$, $Q$, be the parts into which the point of intersection divides the other line, the same process will give

$$(r + d)(r - d) = q \, . \, Q,$$

$$\because qQ = pP.$$

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## PROBLEMS.

1. In the following equations determine which represent circles; find the position of the centre; determine the inclined axes by geometrical construction; and when the numerical value of the radius cannot be exactly ascertained, transform it so that its linear magnitude may be found by a geometrical process.

| | |
|---|---|
| $x^2 + y^2 = 5y - 6x - 3.$ | $x = 4 \pm \sqrt{(6y + 43 - y^2)}.$ |
| $8y - 17 + y^2 = x(x + 2).$ | $(x + a)^2 + y^2 = 6x + cy - d^2.$ |
| $x^2 + 3x = (y - 2)^2 + 2.$ | $2x^2 + 5x = (3 - 2y)y.$ |
| $y^2 - y = 32 - x(20 + x).$ | $x^2 + (y^2 + 1) = 2(2x - 5y).$ |
| $8y + 48 = x^2 + (y - 4)^2.$ | $10x - 8y = 8 - (x^2 + y^2).$ |
| $x^2 - 3\frac{1}{2} - 7y = 9x - y^2.$ | $x^2 + e^2 + 4bx = by - (y + b)^2.$ |
| $(x - c)^2 + (y + a)^2 = f^2 - cy - ax.$ | $x^2 + y - 5 = 3(x - 2) - y^2.$ |
| $2(x^2 + y^2) = 3x(y - 1) + y - 5.$ | $x^2 - 17 + y^2 = 2(-y - 9x).$ |
| $x^2 + 45\frac{1}{4} = 15x - 16y - y^2.$ | $y^2 - 4xy + 3 = x(x - 2y + 1).$ |

$(x+2)^2 = y^2 + 4x - 4y.$ $\quad$ $7 + 5x^2 + 6xy = 4y^2 + y(5+y) + x.$

$(x+3)^2 + (y-2)^2 = 2x - 4y.$ $\quad$ $x(x+10) = 3xy - y^2 - y + 4.$

$x(9x+2y) + 2 = 9y(1-y) + x.$ $\quad$ $3(x^2 + xy) = y(1-3y) + x - 2.$

$6x(1-y) + 5y^2 = 2y - 5x^2.$ $\quad$ $7(x - y^2) - y = x(5x+1) - 2.$

$3(x^2+y^2) - 5xy = (x+y+1)\,10.$ $\quad$ $x(3y-2x) + 2y^2 = y(4y+1) - 6.$

$(x+a)^2 + (y-b)^2 = (2x-b)^2 + (2y+a)^2.$ $\quad$ $(a+b)^2 = (y-b)^2 + x(x-c).$

In the following the axes are supposed rectangular, and the centre of the circle the origin, unless it be otherwise enunciated.

2. The co-ordinates of the centre being $(-a, d)$, find the equation to the tangent.

3. A circle (radius $= r$) touches the axis of $y$ in $d$; find when a tangent at the extremity of a horizontal diameter and another passing through the origin intersect.

4. $2a$ is the base of a triangle, whose vertical angle $= a$ is $< 90°$; find the radius of the circumscribed circle.

5. A particle moves in such a manner, that if at any time the abscissa be increased by three units, and the sum be multiplied by the number of units in the abscissa, the product equals the product of the ordinate into its difference from ten units; what curve does it describe?

6. The ordinate of a point of contact is $\frac{r}{2}$; where does the tangent cut the co-ordinate axes?

7. Determine the circumstances of the intersection of the line $y = \beta$ with the curve $y^2 = 2ax - x^2$: assuming $\beta$ to vary, shew that when this line becomes a tangent, the point of contact is at the same distance from the axis of $y$ as the centre, and on the same side, whether $\beta$ be positive or negative.

8. The co-ordinates of the centre of a circle are $(a, b)$, and $(k, h)$ the point of contact; find where the tangent cuts the axis of $y$.

9. $r$ is the radius of a circle, and a line is drawn from a point in the axis of $x$ to touch the circle; determine the point of contact, and the point in the axis of $x$, when the tangent intercepts a portion of the positive axis of $y$ equal to the diameter of the circle.

10. Two lines revolve about two given points, intersecting at a constant angle; shew that the locus of the point of intersection is a circle.

11. Find the equation to the tangent to a circle, the centre of which is the axis of $y$.

12. In a curve, the sum of the square of the radius vector and twice the square of the ordinate equals the rectangle under the sum of the ordinate and abscissa, and their difference increased by three units; determine the curve. Does it cut the axes?

13. Shew that the angle in a semi-circle is a right angle.

14. $(h, k)$ is the vertex of a triangle circumscribed about a circle, the centre being the origin, and the co-ordinate axes being parallel to two of the sides; determine its area.

15. $(o, c)$ is the centre of a circle; find the equation to the tangent touching it in $(\alpha, \beta)$.

16. A line through $(h, k)$ and the origin is perpendicular to a chord passing through $(\alpha, \beta)$ in the circumference; determine the length of the chord and the radius, the co-ordinates of the centre being $(d, e)$.

17. A particle moves about the origin in such a manner that the cotangent of the inclination of the radius vector at any time is equal to a fraction, of which the numerator is the excess of the ordinate above twelve units, and the denominator the excess of five units above the abscissa; determine the curve.

18. $(h, k)$ being a point from which two tangents are drawn to a circle, and $(a, b)$ one point of contact; determine the radius and the other point of contact, its ordinate having to the radius the ratio $1 : p$.

19. $(o, c)$ being the co-ordinates of the centre of a circle, find the equation to a chord passing through the points to which tangents are drawn from $(a, b)$.

20. The co-ordinates of the extremities of a chord are $(o, o)$, $(a, b)$, and $b > a$; assuming the properties of the circle, find the angle in the larger segment, and prove *Euclid* 32, 3.

21. A tangent is drawn from $(h, k)$, the centre of the circle being the origin: shew that $(h, k)$ is an external point.

## Answers to the Loci.

(1) $x^2 + y\{y + 2\sqrt{(xy)}\} = (m^2 - 1)\,xy$, (2) $d^2 = y^2 + (x - \frac{1}{2}a)^2$.

(3) $(1 - n^2)^2 y^2 + (1 - n^2)\,x + d = n^2 d^2$.

(4) $ax = \sqrt{(x^2 + y^2)}\,(x^2 - y^2)$. (5) $2\,(x \pm a)\,y = a\,(a \pm 2x)$.

(6) $y^2\,(x^2 + y^2)\,(1 - p) = x^2\,(l^2 - x^2 - y^2)$.

(7) $y\,(1{+}n) = \tan a . x$. (8) $ny = a - x$.

(9) $x^2 = \{a \pm \sqrt{(a^2 - 4yl)}\}\;\{a \pm \sqrt{(a^2 - 4yl)} - y\}$.

(11) $y^2 = x(a - x)$. (12) $x^2 = ys - y^2$. (13) $y\sqrt{(s^2 - x^2)} = x^2$.

(14) $(a^2 + b^2)\,(x - b)^2 = x^2\,\{(x - b)^2 + (y - a)^2\}$.

(15) $a^2 + b^2 + 2ab\cos a = x^2 + (y + s)^2$. (17) $x + y = p \pm \sqrt{(p^2 + y^2)}$.

(18) $y^2 = a^2 - x^2$. (19) $a^2 y^2 = b^2 (a^2 - x^2)$.

(20) $x^2 + y^2 - a\sqrt{(x^2 + y^2)} = nya$.

(21) $(a - c)^2\,y^2 = (a + c)^2\,\{(a - c)^2 - x^2\}$.

(22) Page 27. (23) $y^2 = x^2\,(a - x)^2$.

(24) Page 25. (25) Page 24.

(27) $4y^2 = a^2 - (2x - b)^2$.

(28) $y\,[(l + c)\,y + 2c\sqrt{\{(l + c)^2 - x^2\}}] = h^2(l + c) - (x - d)\{x\,(l - c) - ld\}$.

(29) $ay^2 = 2bx\sqrt{(a^2 - x^2)}$. (30) $2by + 2ax = a^2 + b^2$.

(31) $(n + 1)^2\,y^2 = b^2 - x^2$. (32) $y\,(a - x) = \frac{1}{2}.(2ax - x^2)$.

(33) $y$ = infinity when $x = a$.

(34) $x^2 + y^2 = na\,(y - x)$. (35) $(a^2 - y^2)\,x^2 = y^4$.

(36) $(x^4 - 1)\,(m^2 - 1)\,y^2 = ma^2 - (m^2 - 1)\,x^2$.

(37) $y = \tan a\sqrt{(a^2\cos^2 a + x^2)}$. (38) When $y = 0$ $x$ is impossible.

(39) $y\,(x - y) = \frac{1}{4}\,(5 - x^2)\,(x + y)^2$.

(40) $y^2 = x\,(3x - 2b)$. (41) $y = \pm\,(a + b)$.

(42) $y^2(l^2 - x^2) = (2a - x)^2\,(2l^2 - x^2)$. (43) $y^2\,(4b^2 - x^2) = (a^2 + b^2 - y^2)$.

(44) $\{2ax - (2a - y)\,b\}^2 + y^2 = 4a^2$. (45) $y = 2a$ or $3d$.

(46) $2ay = x^2 - a^2$. (47) $2y = x\,\{np \pm \sqrt{(n^2p^2 - 4)}\}$.

(48) Not different signs. (49) $(1 - p^2)\,y = \pm\,x\,\{\sqrt{(p^2 + m^2 - 1)} \mp mp\}$.

(50) $\sqrt{(p^2 + m^2 - 1)}$ would be impossible with some values of $m$ and $p$. (51) different signs if $1 >$ or $< p^2$.

(52) $n^2y^2 + \{a\,(n - 1) - nx\}^2 = a^2$. (53) $y = \pm\sqrt{\{p^2x^2 - (x - d)^2\}}$.

(54) This depends on a transformation.

(55) $y\,(2d - x) = (y - x)^2 + d^2$.

(56) Substitute $\{(x - a), y - b\}$ for $(x, y)$.

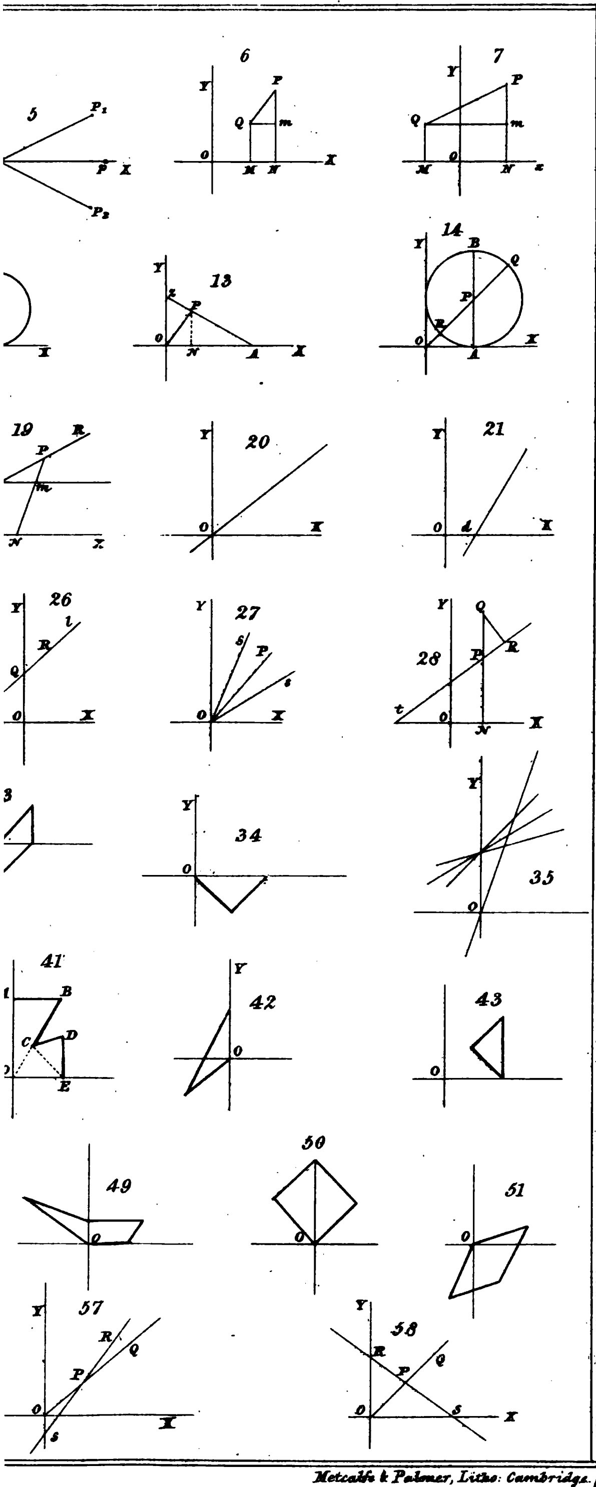
(57) $y^2 = 2\,(s + x)\,\{s - \sqrt{(s^2 - x^2)}\}$.

(58) $y^2\,(n^2 - 1)^2 + \{(n^2 - 1)\,x + a\}^2 = n^2a^2$.

(59) $a^2 = (y + s)^2 + x^2$. (61) Page (26).

THE END.

PLATE 1.

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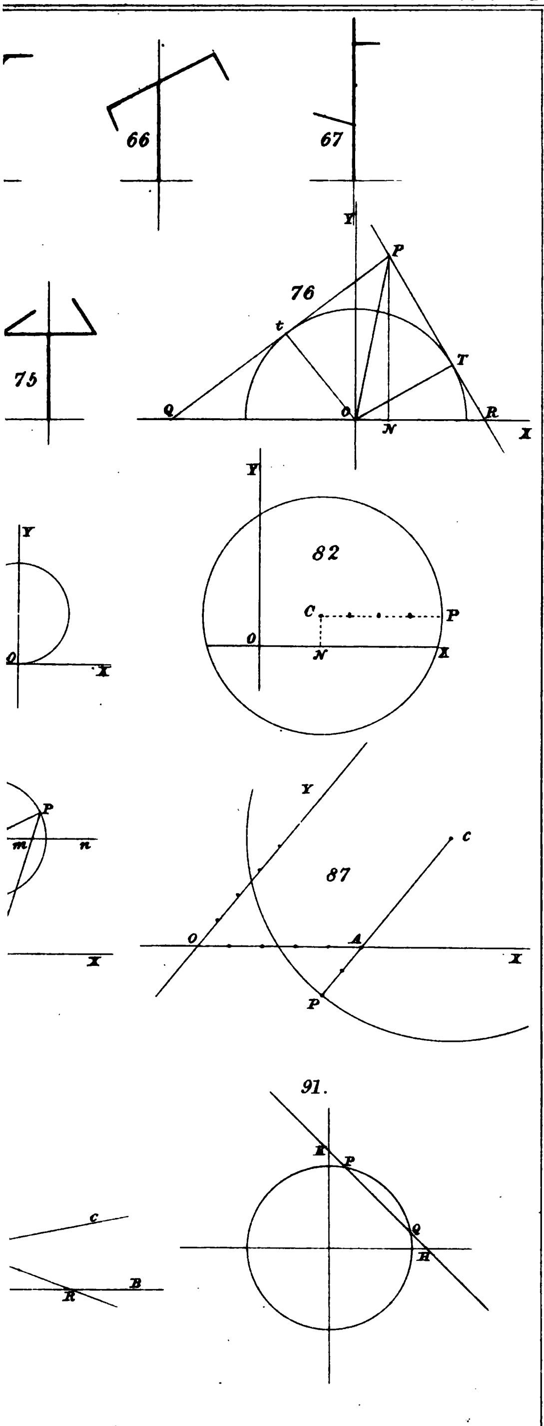


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